

Chapter 11

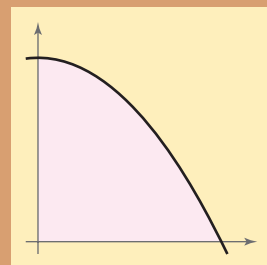
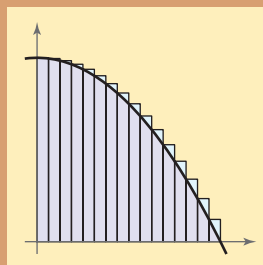
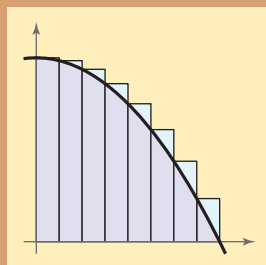
Limits and an Introduction to Calculus

- 11.1 Introduction to Limits
- 11.2 Techniques for Evaluating Limits
- 11.3 The Tangent Line Problem
- 11.4 Limits at Infinity and Limits of Sequences
- 11.5 The Area Problem

Selected Applications

Limit concepts have many real-life applications. The applications listed below represent a small sample of the applications in this chapter.

- Free-Falling Object, Exercises 77 and 78, page 799
- Communications, Exercise 80, page 800
- Market Research, Exercise 64, page 809
- Rate of Change, Exercise 65, page 809
- Average Cost, Exercise 54, page 818
- School Enrollment, Exercise 55, page 818
- Geometry, Exercise 45, page 827



The limit process is a fundamental concept of calculus. In Chapter 11, you will learn many properties of limits and how the limit process can be used to find areas of regions bounded by the graphs of functions. You will also learn how the limit process can be used to find slopes of tangent lines to graphs.

Inga Spence/Index Stock



Americans produce over 200 million pounds of waste each year. Many residents and businesses recycle about 28% of the waste produced. Limits can be used to determine the average cost of recycling material as the amount of material increases infinitely.

11.1 Introduction to Limits

The Limit Concept

The notion of a limit is a *fundamental* concept of calculus. In this chapter, you will learn how to evaluate limits and how they are used in the two basic problems of calculus: the tangent line problem and the area problem.

Example 1 Finding a Rectangle of Maximum Area



You are given 24 inches of wire and are asked to form a rectangle whose area is as large as possible. What dimensions should the rectangle have?

Solution

Let w represent the width of the rectangle and let l represent the length of the rectangle. Because

$$2w + 2l = 24 \quad \text{Perimeter is 24.}$$

it follows that $l = 12 - w$, as shown in Figure 11.1. So, the area of the rectangle is

$$\begin{aligned} A &= lw && \text{Formula for area} \\ &= (12 - w)w && \text{Substitute } 12 - w \text{ for } l. \\ &= 12w - w^2. && \text{Simplify.} \end{aligned}$$

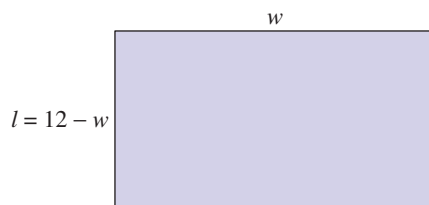


Figure 11.1

Using this model for area, you can experiment with different values of w to see how to obtain the maximum area. After trying several values, it appears that the maximum area occurs when $w = 6$, as shown in the table.

Width, w	5.0	5.5	5.9	6.0	6.1	6.5	7.0
Area, A	35.00	35.75	35.99	36.00	35.99	35.75	35.00

In limit terminology, you can say that “the limit of A as w approaches 6 is 36.” This is written as

$$\lim_{w \rightarrow 6} A = \lim_{w \rightarrow 6} (12w - w^2) = 36.$$

CHECKPOINT Now try Exercise 1.

What you should learn

- Use the definition of a limit to estimate limits.
- Determine whether limits of functions exist.
- Use properties of limits and direct substitution to evaluate limits.

Why you should learn it

The concept of a limit is useful in applications involving maximization. For instance, in Exercise 1 on page 788, the concept of a limit is used to verify the maximum volume of an open box.



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Definition of Limit

Definition of Limit

If $f(x)$ becomes arbitrarily close to a unique number L as x approaches c from either side, the **limit** of $f(x)$ as x approaches c is L . This is written as

$$\lim_{x \rightarrow c} f(x) = L.$$

Example 2 Estimating a Limit Numerically

Use a table to estimate numerically the limit: $\lim_{x \rightarrow 2} (3x - 2)$.

Solution

Let $f(x) = 3x - 2$. Then construct a table that shows values of $f(x)$ for two sets of x -values—one set that approaches 2 from the left and one that approaches 2 from the right.

x	1.9	1.99	1.999	2.0	2.001	2.01	2.1
$f(x)$	3.700	3.970	3.997	?	4.003	4.030	4.300

From the table, it appears that the closer x gets to 2, the closer $f(x)$ gets to 4. So, you can estimate the limit to be 4. Figure 11.2 adds further support to this conclusion.

CHECKPOINT Now try Exercise 3.

In Figure 11.2, note that the graph of $f(x) = 3x - 2$ is continuous. For graphs that are not continuous, finding a limit can be more difficult.

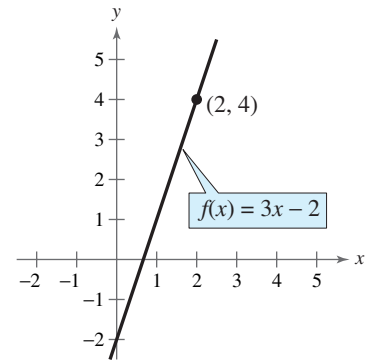


Figure 11.2

Example 3 Estimating a Limit Numerically

Use a table to estimate numerically the limit: $\lim_{x \rightarrow 0} \frac{x}{\sqrt{x+1}-1}$.

Solution

Let $f(x) = \frac{x}{\sqrt{x+1}-1}$. Then construct a table that shows values of $f(x)$ for two sets of x -values—one set that approaches 0 from the left and one that approaches 0 from the right.

x	-0.01	-0.001	-0.0001	0	0.0001	0.001	0.01
$f(x)$	1.99499	1.99949	1.99995	?	2.00005	2.00050	2.00499

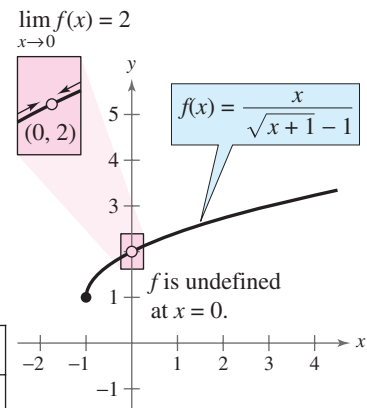


Figure 11.3

From the table, it appears that the limit is 2. The graph shown in Figure 11.3 verifies that the limit is 2.

CHECKPOINT Now try Exercise 5.

In Example 3, note that $f(x)$ has a limit when $x \rightarrow 0$ even though the function is not defined when $x = 0$. This often happens, and it is important to realize that *the existence or nonexistence of $f(x)$ at $x = c$ has no bearing on the existence of the limit of $f(x)$ as x approaches c .*

Example 4 Using a Graphing Utility to Estimate a Limit

Estimate the limit: $\lim_{x \rightarrow 1} \frac{x^3 - x^2 + x - 1}{x - 1}$.

Numerical Solution

Let $f(x) = (x^3 - x^2 + x - 1)/(x - 1)$. Because you are finding the limit of $f(x)$ as x approaches 1, use the *table* feature of a graphing utility to create a table that shows the value of the function for x starting at $x = 0.9$ and setting the table step to 0.01, as shown in Figure 11.4(a). Then change the table so that x starts at 0.99 and set the table step to 0.001, as shown in Figure 11.4(b). From the tables, you can estimate the limit to be 2.

X	Y1
.97	1.9409
.98	1.9604
.99	1.9801
.99	ERROR
1.01	2.0201
1.02	2.0404
1.03	2.0609

(a)
Figure 11.4

X	Y1
.997	1.994
.998	1.996
.999	1.998
.999	ERROR
1.001	2.002
1.002	2.004
1.003	2.006

(b)

 **CHECKPOINT** Now try Exercise 11.

Example 5 Using a Graph to Find a Limit

Find the limit of $f(x)$ as x approaches 3, where f is defined as

$$f(x) = \begin{cases} 2, & x \neq 3 \\ 0, & x = 3 \end{cases}$$

Solution

Because $f(x) = 2$ for all x other than $x = 3$ and because the value of $f(3)$ is immaterial, it follows that the limit is 2 (see Figure 11.6). So, you can write

$$\lim_{x \rightarrow 3} f(x) = 2.$$

The fact that $f(3) = 0$ has no bearing on the existence or value of the limit as x approaches 3. For instance, if the function were defined as

$$f(x) = \begin{cases} 2, & x \neq 3 \\ 4, & x = 3 \end{cases}$$

the limit as x approaches 3 would be the same.

 **CHECKPOINT** Now try Exercise 25.

Graphical Solution

Use a graphing utility to graph $y = (x^3 - x^2 + x - 1)/(x - 1)$ using a *decimal* setting. Then use the *zoom* and *trace* features to determine that as x gets closer and closer to 1, y gets closer and closer to 2 from the left and from the right, as shown in Figure 11.5. Using the *trace* feature, notice that there is no value given for y when $x = 1$, and that there is a hole or break in the graph when $x = 1$.

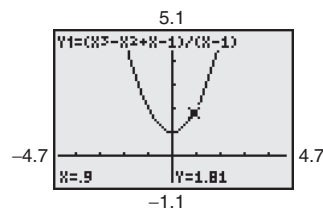


Figure 11.5

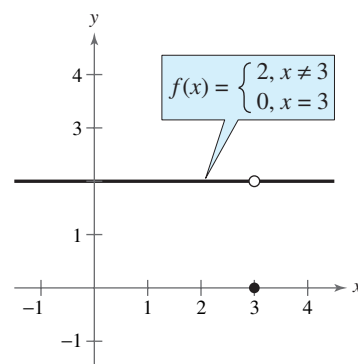


Figure 11.6

Limits That Fail to Exist

Next, you will examine some functions for which limits do not exist.

Example 6 Comparing Left and Right Behavior

Show that the limit does not exist.

$$\lim_{x \rightarrow 0} \frac{|x|}{x}$$

Solution

Consider the graph of the function given by $f(x) = |x|/x$. In Figure 11.7, you can see that for positive x -values

$$\frac{|x|}{x} = 1, \quad x > 0$$

and for negative x -values

$$\frac{|x|}{x} = -1, \quad x < 0.$$

This means that no matter how close x gets to 0, there will be both positive and negative x -values that yield $f(x) = 1$ and $f(x) = -1$. This implies that the limit does not exist.

 **CHECKPOINT** Now try Exercise 31.

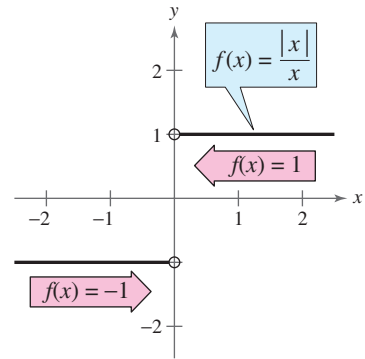


Figure 11.7

Example 7 Unbounded Behavior

Discuss the existence of the limit.

$$\lim_{x \rightarrow 0} \frac{1}{x^2}$$

Solution

Let $f(x) = 1/x^2$. In Figure 11.8, note that as x approaches 0 from either the right or the left, $f(x)$ increases without bound. This means that by choosing x close enough to 0, you can force $f(x)$ to be as large as you want. For instance, $f(x)$ will be larger than 100 if you choose x that is within $\frac{1}{10}$ of 0. That is,

$$0 < |x| < \frac{1}{10} \quad \Rightarrow \quad f(x) = \frac{1}{x^2} > 100.$$

Similarly, you can force $f(x)$ to be larger than 1,000,000, as follows.

$$0 < |x| < \frac{1}{1000} \quad \Rightarrow \quad f(x) = \frac{1}{x^2} > 1,000,000$$

Because $f(x)$ is not approaching a unique real number L as x approaches 0, you can conclude that the limit does not exist.

 **CHECKPOINT** Now try Exercise 32.

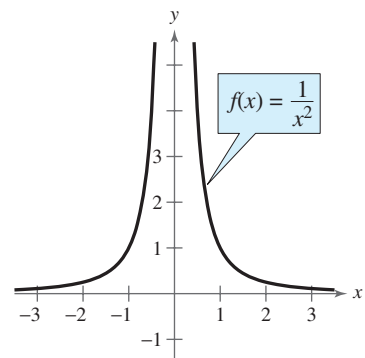


Figure 11.8

Example 8 Oscillating Behavior

Discuss the existence of the limit.

$$\lim_{x \rightarrow 0} \sin\left(\frac{1}{x}\right)$$

Solution

Let $f(x) = \sin(1/x)$. In Figure 11.9, you can see that as x approaches 0, $f(x)$ oscillates between -1 and 1 . Therefore, the limit does not exist because no matter how close you are to 0, it is possible to choose values of x_1 and x_2 such that $\sin(1/x_1) = 1$ and $\sin(1/x_2) = -1$, as indicated in the table.

x	$\frac{2}{\pi}$	$\frac{2}{3\pi}$	$\frac{2}{5\pi}$	$\frac{2}{7\pi}$	$\frac{2}{9\pi}$	$\frac{2}{11\pi}$	$x \rightarrow 0$
$\sin\left(\frac{1}{x}\right)$	1	-1	1	-1	1	-1	Limit does not exist.

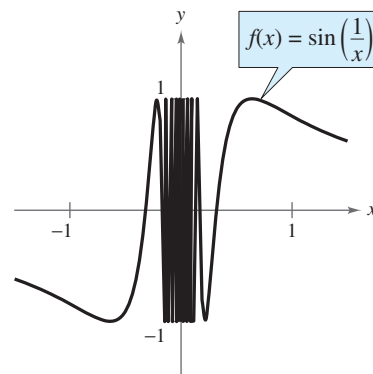


Figure 11.9

CHECKPOINT Now try Exercise 33.

Examples 6, 7, and 8 show three of the most common types of behavior associated with the *nonexistence* of a limit.

Conditions Under Which Limits Do Not Exist

The limit of $f(x)$ as $x \rightarrow c$ does not exist if any of the following conditions is true.

1. $f(x)$ approaches a different number from the right side of c than it approaches from the left side of c . Example 6
2. $f(x)$ increases or decreases without bound as x approaches c . Example 7
3. $f(x)$ oscillates between two fixed values as x approaches c . Example 8

TECHNOLOGY TIP A graphing utility can help you discover the behavior of a function near the x -value at which you are trying to evaluate a limit. When you do this, however, you should realize that you can't always trust the graphs that graphing utilities display. For instance, if you use a graphing utility to graph the function in Example 8 over an interval containing 0, you will most likely obtain an incorrect graph, as shown in Figure 11.10. The reason that a graphing utility can't show the correct graph is that the graph has infinitely many oscillations over any interval that contains 0.

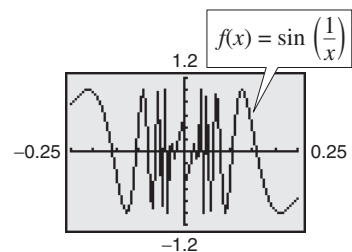


Figure 11.10

Properties of Limits and Direct Substitution

You have seen that sometimes the limit of $f(x)$ as $x \rightarrow c$ is simply $f(c)$. In such cases, it is said that the limit can be evaluated by **direct substitution**. That is,

$$\lim_{x \rightarrow c} f(x) = f(c). \quad \text{Substitute } c \text{ for } x.$$

There are many “well-behaved” functions, such as polynomial functions and rational functions with nonzero denominators, that have this property. Some of the basic ones are included in the following list.

Basic Limits

Let b and c be real numbers and let n be a positive integer.

1. $\lim_{x \rightarrow c} b = b$
2. $\lim_{x \rightarrow c} x = c$
3. $\lim_{x \rightarrow c} x^n = c^n$ (See the proof on page 835.)
4. $\lim_{x \rightarrow c} \sqrt[n]{x} = \sqrt[n]{c}$, for n even and $c > 0$

Trigonometric functions can also be included in this list. For instance,

$$\begin{aligned} \lim_{x \rightarrow \pi} \sin x &= \sin \pi \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} \lim_{x \rightarrow 0} \cos x &= \cos 0 \\ &= 1. \end{aligned}$$

By combining the basic limits with the following operations, you can find limits for a wide variety of functions.

Properties of Limits

Let b and c be real numbers, let n be a positive integer, and let f and g be functions with the following limits.

$$\lim_{x \rightarrow c} f(x) = L \quad \text{and} \quad \lim_{x \rightarrow c} g(x) = K$$

1. Scalar multiple: $\lim_{x \rightarrow c} [bf(x)] = bL$
2. Sum or difference: $\lim_{x \rightarrow c} [f(x) \pm g(x)] = L \pm K$
3. Product: $\lim_{x \rightarrow c} [f(x)g(x)] = LK$
4. Quotient: $\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \frac{L}{K}$, provided $K \neq 0$
5. Power: $\lim_{x \rightarrow c} [f(x)]^n = L^n$

Exploration

Use a graphing utility to graph the tangent function. What are

$$\lim_{x \rightarrow 0} \tan x \quad \text{and} \quad \lim_{x \rightarrow \pi/4} \tan x?$$

What can you say about the existence of the limit

$$\lim_{x \rightarrow \pi/2} \tan x?$$

Example 9 Direct Substitution and Properties of Limits

Find each limit.

- a. $\lim_{x \rightarrow 4} x^2$ b. $\lim_{x \rightarrow 4} 5x$ c. $\lim_{x \rightarrow \pi} \frac{\tan x}{x}$
 d. $\lim_{x \rightarrow 9} \sqrt{x}$ e. $\lim_{x \rightarrow \pi} (x \cos x)$ f. $\lim_{x \rightarrow 3} (x + 4)^2$

Solution

You can use the properties of limits and direct substitution to evaluate each limit.

- a. $\lim_{x \rightarrow 4} x^2 = (4)^2$ Direct Substitution
 $= 16$
- b. $\lim_{x \rightarrow 4} 5x = 5 \lim_{x \rightarrow 4} x$ Scalar Multiple Property
 $= 5(4)$
 $= 20$
- c. $\lim_{x \rightarrow \pi} \frac{\tan x}{x} = \frac{\lim_{x \rightarrow \pi} \tan x}{\lim_{x \rightarrow \pi} x}$ Quotient Property
 $= \frac{0}{\pi} = 0$
- d. $\lim_{x \rightarrow 9} \sqrt{x} = \sqrt{9}$
 $= 3$
- e. $\lim_{x \rightarrow \pi} (x \cos x) = (\lim_{x \rightarrow \pi} x)(\lim_{x \rightarrow \pi} \cos x)$ Product Property
 $= \pi(\cos \pi)$
 $= -\pi$
- f. $\lim_{x \rightarrow 3} (x + 4)^2 = \left[(\lim_{x \rightarrow 3} x) + (\lim_{x \rightarrow 3} 4) \right]^2$ Sum and Power Properties
 $= (3 + 4)^2$
 $= 7^2 = 49$

CHECKPOINT Now try Exercise 49.

TECHNOLOGY TIP When evaluating limits, remember that there are several ways to solve most problems. Often, a problem can be solved *numerically*, *graphically*, or *algebraically*. The limits in Example 9 were found algebraically. You can verify these solutions numerically and/or graphically. For instance, to verify the limit in Example 9(a) numerically, use the *table* feature of a graphing utility to create a table, as shown in Figure 11.11. From the table, you can see that the limit as x approaches 4 is 16. Now, to verify the limit graphically, use a graphing utility to graph $y = x^2$. Using the *zoom* and *trace* features, you can determine that the limit as x approaches 4 is 16, as shown in Figure 11.12.

Exploration

Sketch the graph of each function. Then find the limits of each function as x approaches 1 and as x approaches 2. What conclusions can you make?

a. $f(x) = x + 1$

b. $g(x) = \frac{x^2 - 1}{x - 1}$

c. $h(x) = \frac{x^3 - 2x^2 - x + 2}{x^2 - 3x + 2}$

Use a graphing utility to graph each function above. Does the graphing utility distinguish among the three graphs? Write a short explanation of your findings.

X	Y1	
2.5	6.25	
3.0	9	
3.5	12.25	
4.0	16	
4.5	20.25	
5.0	25	
5.5	30.25	

X=4

Figure 11.11

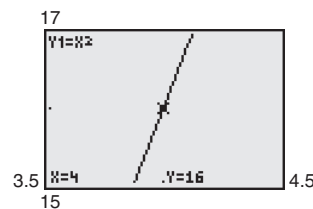


Figure 11.12

The results of using direct substitution to evaluate limits of polynomial and rational functions are summarized as follows.

Limits of Polynomial and Rational Functions

1. If p is a polynomial function and c is a real number, then

$$\lim_{x \rightarrow c} p(x) = p(c). \quad (\text{See the proof on page 835.})$$

2. If r is a rational function given by $r(x) = p(x)/q(x)$, and c is a real number such that $q(c) \neq 0$, then

$$\lim_{x \rightarrow c} r(x) = r(c) = \frac{p(c)}{q(c)}.$$

Example 10 Evaluating Limits by Direct Substitution

Find each limit.

a. $\lim_{x \rightarrow -1} (x^2 + x - 6)$ b. $\lim_{x \rightarrow -1} \frac{x^2 + x - 6}{x + 3}$

Solution

The first function is a polynomial function and the second is a rational function (with a nonzero denominator at $x = -1$). So, you can evaluate the limits by direct substitution.

$$\begin{aligned} \text{a. } \lim_{x \rightarrow -1} (x^2 + x - 6) &= (-1)^2 + (-1) - 6 \\ &= -6 \end{aligned}$$

$$\begin{aligned} \text{b. } \lim_{x \rightarrow -1} \frac{x^2 + x - 6}{x + 3} &= \frac{(-1)^2 + (-1) - 6}{-1 + 3} \\ &= -\frac{6}{2} \\ &= -3 \end{aligned}$$



CHECKPOINT Now try Exercise 53.

Exploration

Use a graphing utility to graph the function

$$f(x) = \frac{x^2 - 3x - 10}{x - 5}.$$

Use the *trace* feature to approximate $\lim_{x \rightarrow 4} f(x)$. What do you think $\lim_{x \rightarrow 5} f(x)$ equals? Is f defined at $x = 5$? Does this affect the existence of the limit as x approaches 5?

11.1 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Vocabulary Check

Fill in the blanks.

- If $f(x)$ becomes arbitrarily close to a unique number L as x approaches c from either side, the _____ of $f(x)$ as x approaches c is L .
- The limit of $f(x)$ as $x \rightarrow c$ does not exist if $f(x)$ _____ between two fixed values.
- To evaluate the limit of a polynomial function, use _____.

1. **Geometry** You create an open box from a square piece of material, 24 centimeters on a side. You cut equal squares from the corners and turn up the sides.

- Draw and label a diagram that represents the box.
- Verify that the volume of the box is given by

$$V = 4x(12 - x)^2.$$

- The box has a maximum volume when $x = 4$. Use a graphing utility to complete the table and observe the behavior of the function as x approaches 4. Use the table to find $\lim_{x \rightarrow 4} V$.

x	3	3.5	4	4.1	4.5	5
V						

- Use a graphing utility to graph the volume function. Verify that the volume is maximum when $x = 4$.

2. **Geometry** You are given wire and are asked to form a right triangle with a hypotenuse of $\sqrt{18}$ inches whose area is as large as possible.

- Draw and label a diagram that shows the base x and height y of the triangle.
- Verify that the area of the triangle is given by

$$A = \frac{1}{2}x\sqrt{18 - x^2}.$$

- The triangle has a maximum area when $x = 3$ inches. Use a graphing utility to complete the table and observe the behavior of the function as x approaches 3. Use the table to find $\lim_{x \rightarrow 3} A$.

x	2	2.5	2.9	3	3.1	3.5	4
A							

- Use a graphing utility to graph the area function. Verify that the area is maximum when $x = 3$ inches.

In Exercises 3–10, complete the table and use the result to estimate the limit numerically. Determine whether or not the limit can be reached.

3. $\lim_{x \rightarrow 2} (5x + 4)$

x	1.9	1.99	1.999	2	2.001	2.01	2.1
$f(x)$?			

4. $\lim_{x \rightarrow 1} (2x^2 + x - 4)$

x	0.9	0.99	0.999	1	1.001	1.01	1.1
$f(x)$?			

5. $\lim_{x \rightarrow 3} \frac{x - 3}{x^2 - 9}$

x	2.9	2.99	2.999	3	3.001	3.01	3.1
$f(x)$?			

6. $\lim_{x \rightarrow -1} \frac{x + 1}{x^2 - x - 2}$

x	-1.1	-1.01	-1.001	-1	-0.999
$f(x)$?	

x	-0.99	-0.9
$f(x)$		

7. $\lim_{x \rightarrow 0} \frac{\sin 2x}{x}$

x	-0.1	-0.01	-0.001	0	0.001
$f(x)$?	

x	0.01	0.1
$f(x)$		

8. $\lim_{x \rightarrow 0} \frac{\tan x}{2x}$

x	-0.1	-0.01	-0.001	0	0.001
$f(x)$?	

x	0.01	0.1
$f(x)$		

9. $\lim_{x \rightarrow 0} \frac{e^{2x} - 1}{x}$

x	-0.1	-0.01	-0.001	0	0.001
$f(x)$?	

x	0.01	0.1
$f(x)$		

10. $\lim_{x \rightarrow 1} \frac{\ln x}{x - 1}$

x	0.9	0.99	0.999	1	1.001	1.01	1.1
$f(x)$?			

In Exercises 11–24, use the *table* feature of a graphing utility to create a table for the function and use the result to estimate the limit numerically. Use the graphing utility to graph the corresponding function to confirm your result graphically.

11. $\lim_{x \rightarrow 1} \frac{x - 1}{x^2 + 2x - 3}$

13. $\lim_{x \rightarrow 0} \frac{\sqrt{x + 5} - \sqrt{5}}{x}$

15. $\lim_{x \rightarrow -4} \frac{\frac{x}{x + 2} - 2}{x + 4}$

17. $\lim_{x \rightarrow 0} \frac{\sin x}{x}$

19. $\lim_{x \rightarrow 0} \frac{\sin^2 x}{x}$

21. $\lim_{x \rightarrow 0} \frac{e^{2x} - 1}{2x}$

23. $\lim_{x \rightarrow 1} \frac{\ln(2x - 1)}{x - 1}$

12. $\lim_{x \rightarrow -2} \frac{x + 2}{x^2 + 5x + 6}$

14. $\lim_{x \rightarrow -3} \frac{\sqrt{1 - x} - 2}{x + 3}$

16. $\lim_{x \rightarrow 2} \frac{\frac{1}{x + 2} - \frac{1}{4}}{x - 2}$

18. $\lim_{x \rightarrow 0} \frac{\cos x - 1}{x}$

20. $\lim_{x \rightarrow 0} \frac{2x}{\tan 4x}$

22. $\lim_{x \rightarrow 0} \frac{1 - e^{-4x}}{x}$

24. $\lim_{x \rightarrow 1} \frac{\ln(x^2)}{x - 1}$

In Exercises 25–28, graph the function and find the limit (if it exists) as x approaches 2.

25. $f(x) = \begin{cases} 2x + 1, & x < 2 \\ x + 3, & x \geq 2 \end{cases}$

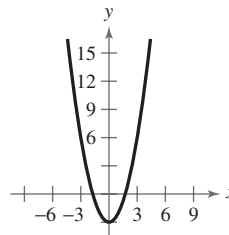
26. $f(x) = \begin{cases} 8 - x^2, & x \leq 2 \\ x + 2, & x > 2 \end{cases}$

27. $f(x) = \begin{cases} 2x + 1, & x \leq 2 \\ x + 4, & x > 2 \end{cases}$

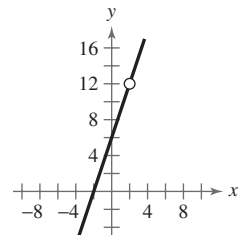
28. $f(x) = \begin{cases} -2x, & x \leq 2 \\ x^2 - 4x + 1, & x > 2 \end{cases}$

In Exercises 29–36, use the graph to find the limit (if it exists). If the limit does not exist, explain why.

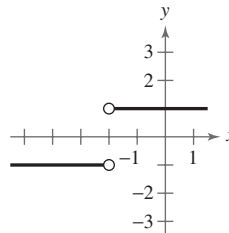
29. $\lim_{x \rightarrow -4} (x^2 - 3)$



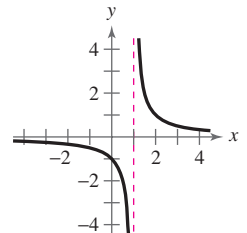
30. $\lim_{x \rightarrow 2} \frac{3x^2 - 12}{x - 2}$



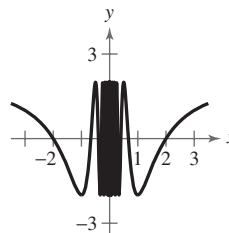
31. $\lim_{x \rightarrow -2} \frac{|x + 2|}{x + 2}$



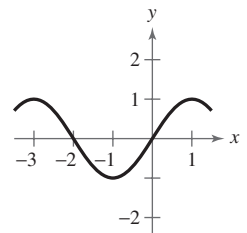
32. $\lim_{x \rightarrow 1} \frac{1}{x - 1}$



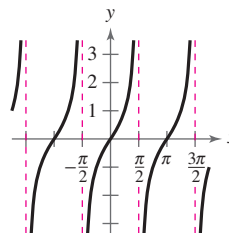
33. $\lim_{x \rightarrow 0} 2 \cos \frac{\pi}{x}$



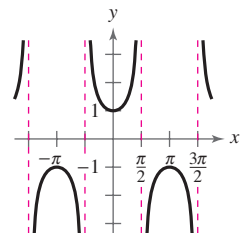
34. $\lim_{x \rightarrow -1} \sin \frac{\pi x}{2}$



35. $\lim_{x \rightarrow \pi/2} \tan x$



36. $\lim_{x \rightarrow \pi/2} \sec x$



In Exercises 37–46, use a graphing utility to graph the function and use the graph to determine whether or not the limit exists. If the limit does not exist, explain why.

37. $f(x) = \frac{5}{2 + e^{1/x}}, \lim_{x \rightarrow 0} f(x)$

38. $f(x) = \frac{e^x - 1}{x}, \lim_{x \rightarrow 0} f(x)$

39. $f(x) = \cos \frac{1}{x}, \lim_{x \rightarrow 0} f(x)$

40. $f(x) = \sin \pi x, \lim_{x \rightarrow -1} f(x)$

41. $f(x) = \frac{\sqrt{x+3} - 1}{x - 4}, \lim_{x \rightarrow 4} f(x)$

42. $f(x) = \frac{\sqrt{x+5} - 4}{x - 2}, \lim_{x \rightarrow 2} f(x)$

43. $f(x) = \frac{x - 1}{x^2 - 4x + 3}, \lim_{x \rightarrow 1} f(x)$

44. $f(x) = \frac{7}{x - 3}, \lim_{x \rightarrow 3} f(x)$

45. $f(x) = \ln(x + 3), \lim_{x \rightarrow 4} f(x)$

46. $f(x) = \ln(7 - x), \lim_{x \rightarrow -1} f(x)$

In Exercises 47 and 48, use the given information to evaluate each limit.

47. $\lim_{x \rightarrow c} f(x) = 3, \lim_{x \rightarrow c} g(x) = 6$

(a) $\lim_{x \rightarrow c} [-2g(x)]$ (b) $\lim_{x \rightarrow c} [f(x) + g(x)]$

(c) $\lim_{x \rightarrow c} \frac{f(x)}{g(x)}$ (d) $\lim_{x \rightarrow c} \sqrt{f(x)}$

48. $\lim_{x \rightarrow c} f(x) = 5, \lim_{x \rightarrow c} g(x) = -2$

(a) $\lim_{x \rightarrow c} [f(x) + g(x)]^2$ (b) $\lim_{x \rightarrow c} [6f(x)g(x)]$

(c) $\lim_{x \rightarrow c} \frac{5g(x)}{4f(x)}$ (d) $\lim_{x \rightarrow c} \frac{1}{\sqrt{f(x)}}$

In Exercises 49 and 50, find (a) $\lim_{x \rightarrow 2} f(x)$, (b) $\lim_{x \rightarrow 2} g(x)$, (c) $\lim_{x \rightarrow 2} [f(x)g(x)]$, and (d) $\lim_{x \rightarrow 2} [g(x) - f(x)]$.

49. $f(x) = x^3, g(x) = \frac{\sqrt{x^2 + 5}}{2x^2}$

50. $f(x) = \frac{x}{3 - x}, g(x) = \sin \pi x$

In Exercises 51–70, find the limit by direct substitution.

51. $\lim_{x \rightarrow 5} (10 - x^2)$

52. $\lim_{x \rightarrow -2} (\frac{1}{2}x^3 - 5x)$

53. $\lim_{x \rightarrow -3} (2x^2 + 4x + 1)$

54. $\lim_{x \rightarrow -2} (x^3 - 6x + 5)$

55. $\lim_{x \rightarrow 3} \left(-\frac{9}{x}\right)$

56. $\lim_{x \rightarrow -5} \frac{6}{x + 2}$

57. $\lim_{x \rightarrow -3} \frac{3x}{x^2 + 1}$

58. $\lim_{x \rightarrow 4} \frac{x - 1}{x^2 + 2x + 3}$

59. $\lim_{x \rightarrow -2} \frac{5x + 3}{2x - 9}$

61. $\lim_{x \rightarrow -1} \sqrt{x + 2}$

63. $\lim_{x \rightarrow 7} \frac{5x}{\sqrt{x + 2}}$

65. $\lim_{x \rightarrow 3} e^x$

67. $\lim_{x \rightarrow \pi} \sin 2x$

69. $\lim_{x \rightarrow 1/2} \arcsin x$

60. $\lim_{x \rightarrow 3} \frac{x^2 + 1}{x}$

62. $\lim_{x \rightarrow 3} \sqrt[3]{x^2 - 1}$

64. $\lim_{x \rightarrow 8} \frac{\sqrt{x + 1}}{x - 4}$

66. $\lim_{x \rightarrow e} \ln x$

68. $\lim_{x \rightarrow \pi} \tan x$

70. $\lim_{x \rightarrow 1} \arccos \frac{x}{2}$

Synthesis

True or False? In Exercises 71 and 72, determine whether the statement is true or false. Justify your answer.

71. The limit of a function as x approaches c does not exist if the function approaches -3 from the left of c and 3 from the right of c .

72. The limit of the product of two functions is equal to the product of the limits of the two functions.

73. **Think About It** From Exercises 3 to 10, select a limit that can be reached and one that cannot be reached.

(a) Use a graphing utility to graph the corresponding functions using a standard viewing window. Do the graphs reveal whether or not the limit can be reached? Explain.

(b) Use a graphing utility to graph the corresponding functions using a *decimal* setting. Do the graphs reveal whether or not the limit can be reached? Explain.

74. **Think About It** Use the results of Exercise 73 to draw a conclusion as to whether or not you can use the graph generated by a graphing utility to determine reliably if a limit can be reached.

75. **Think About It**

(a) If $f(2) = 4$, can you conclude anything about $\lim_{x \rightarrow 2} f(x)$? Explain your reasoning.

(b) If $\lim_{x \rightarrow 2} f(x) = 4$, can you conclude anything about $f(2)$? Explain your reasoning.

76. **Writing** Write a brief description of the meaning of the notation $\lim_{x \rightarrow 5} f(x) = 12$.

Skills Review

In Exercises 77–82, simplify the rational expression.

77. $\frac{5 - x}{3x - 15}$

78. $\frac{x^2 - 81}{9 - x}$

79. $\frac{15x^2 + 7x - 4}{15x^2 + x - 2}$

80. $\frac{x^2 - 12x + 36}{x^2 - 7x + 6}$

81. $\frac{x^3 + 27}{x^2 + x - 6}$

82. $\frac{x^3 - 8}{x^2 - 4}$

11.2 Techniques for Evaluating Limits

Dividing Out Technique

In Section 11.1, you studied several types of functions whose limits can be evaluated by direct substitution. In this section, you will study several techniques for evaluating limits of functions for which direct substitution fails.

Suppose you were asked to find the following limit.

$$\lim_{x \rightarrow -3} \frac{x^2 + x - 6}{x + 3}$$

Direct substitution fails because -3 is a zero of the denominator. By using a table, however, it appears that the limit of the function as $x \rightarrow -3$ is -5 .

x	-3.01	-3.001	-3.0001	-3	-2.9999	-2.999	-2.99
$\frac{x^2 + x - 6}{x + 3}$	-5.01	-5.001	-5.0001	$?$	-4.9999	-4.999	-4.99

Another way to find the limit of this function is shown in Example 1.

Example 1 Dividing Out Technique

Find the limit: $\lim_{x \rightarrow -3} \frac{x^2 + x - 6}{x + 3}$.

Solution

Begin by factoring the numerator and dividing out any common factors.

$$\begin{aligned} \lim_{x \rightarrow -3} \frac{x^2 + x - 6}{x + 3} &= \lim_{x \rightarrow -3} \frac{(x - 2)(x + 3)}{x + 3} && \text{Factor numerator.} \\ &= \lim_{x \rightarrow -3} \frac{(x - 2)\cancel{(x + 3)}}{\cancel{x + 3}} && \text{Divide out common factor.} \\ &= \lim_{x \rightarrow -3} (x - 2) && \text{Simplify.} \\ &= -3 - 2 && \text{Direct substitution} \\ &= -5 && \text{Simplify.} \end{aligned}$$



Now try Exercise 7.

This procedure for evaluating a limit is called the **dividing out technique**. The validity of this technique stems from the fact that if two functions agree at all but a single number c , they must have identical limit behavior at $x = c$. In Example 1, the functions given by

$$f(x) = \frac{x^2 + x - 6}{x + 3} \quad \text{and} \quad g(x) = x - 2$$

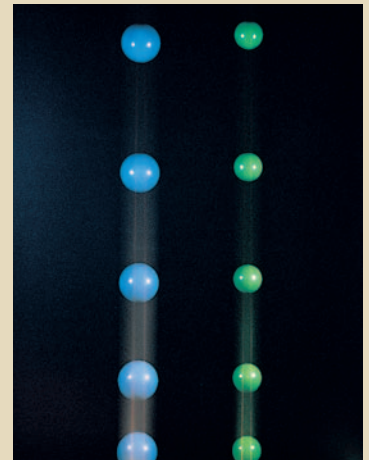
agree at all values of x other than $x = -3$. So, you can use $g(x)$ to find the limit of $f(x)$.

What you should learn

- Use the dividing out technique to evaluate limits of functions.
- Use the rationalizing technique to evaluate limits of functions.
- Approximate limits of functions graphically and numerically.
- Evaluate one-sided limits of functions.
- Evaluate limits of difference quotients from calculus.

Why you should learn it

Many definitions in calculus involve the limit of a function. For instance, in Exercises 77 and 78 on page 799, the definition of the velocity of a free-falling object at any instant in time involves finding the limit of a position function.



Peticolas Megna/Fundamental Photographs

Prerequisite Skills

To review factoring techniques, see Appendix G, Study Capsule 1.

The dividing out technique should be applied only when direct substitution produces 0 in both the numerator *and* the denominator. The resulting fraction, $\frac{0}{0}$, has no meaning as a real number. It is called an **indeterminate form** because you cannot, from the form alone, determine the limit. When you try to evaluate a limit of a rational function by direct substitution and encounter this form, you can conclude that the numerator and denominator must have a common factor. After factoring and dividing out, you should try direct substitution again.

Example 2 Dividing Out Technique

Find the limit.

$$\lim_{x \rightarrow 1} \frac{x - 1}{x^3 - x^2 + x - 1}$$

Solution

Begin by substituting $x = 1$ into the numerator and denominator.

$$1 - 1 = 0$$

Numerator is 0 when $x = 1$.

$$1^3 - 1^2 + 1 - 1 = 0$$

Denominator is 0 when $x = 1$.

Because both the numerator and denominator are zero when $x = 1$, direct substitution will not yield the limit. To find the limit, you should factor the numerator and denominator, divide out any common factors, and then try direct substitution again.

$$\lim_{x \rightarrow 1} \frac{x - 1}{x^3 - x^2 + x - 1} = \lim_{x \rightarrow 1} \frac{x - 1}{(x - 1)(x^2 + 1)}$$

Factor denominator.

$$= \lim_{x \rightarrow 1} \frac{\cancel{x - 1}}{(\cancel{x - 1})(x^2 + 1)}$$

Divide out common factor.

$$= \lim_{x \rightarrow 1} \frac{1}{x^2 + 1}$$

Simplify.

$$= \frac{1}{1^2 + 1}$$

Direct substitution

$$= \frac{1}{2}$$

Simplify.

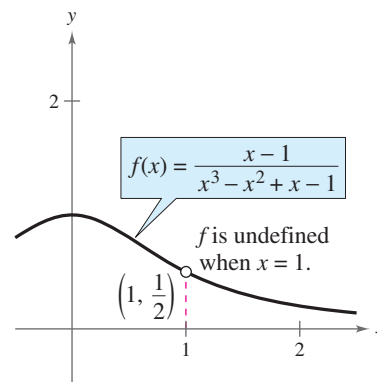


Figure 11.13

This result is shown graphically in Figure 11.13.

CHECKPOINT Now try Exercise 9.

In Example 2, the factorization of the denominator can be obtained by dividing by $(x - 1)$ or by grouping as follows.

$$\begin{aligned} x^3 - x^2 + x - 1 &= x^2(x - 1) + (x - 1) \\ &= (x - 1)(x^2 + 1) \end{aligned}$$

Rationalizing Technique

Another way to find the limits of some functions is first to rationalize the numerator of the function. This is called the **rationalizing technique**. Recall that rationalizing the numerator means multiplying the numerator and denominator by the conjugate of the numerator. For instance, the conjugate of $\sqrt{x} + 4$ is $\sqrt{x} - 4$.

Example 3 Rationalizing Technique

Find the limit: $\lim_{x \rightarrow 0} \frac{\sqrt{x+1} - 1}{x}$.

Solution

By direct substitution, you obtain the indeterminate form $\frac{0}{0}$.

$$\lim_{x \rightarrow 0} \frac{\sqrt{x+1} - 1}{x} = \frac{\sqrt{0+1} - 1}{0} = \frac{0}{0} \quad \text{Indeterminate form}$$

In this case, you can rewrite the fraction by rationalizing the numerator.

$$\begin{aligned} \frac{\sqrt{x+1} - 1}{x} &= \left(\frac{\sqrt{x+1} - 1}{x} \right) \left(\frac{\sqrt{x+1} + 1}{\sqrt{x+1} + 1} \right) \\ &= \frac{(x+1) - 1}{x(\sqrt{x+1} + 1)} && \text{Multiply.} \\ &= \frac{x}{x(\sqrt{x+1} + 1)} && \text{Simplify.} \\ &= \frac{\cancel{x}}{\cancel{x}(\sqrt{x+1} + 1)} && \text{Divide out common factor.} \\ &= \frac{1}{\sqrt{x+1} + 1}, \quad x \neq 0 && \text{Simplify.} \end{aligned}$$

Now you can evaluate the limit by direct substitution.

$$\lim_{x \rightarrow 0} \frac{\sqrt{x+1} - 1}{x} = \lim_{x \rightarrow 0} \frac{1}{\sqrt{x+1} + 1} = \frac{1}{\sqrt{0+1} + 1} = \frac{1}{1+1} = \frac{1}{2}$$

You can reinforce your conclusion that the limit is $\frac{1}{2}$ by constructing a table, as shown below, or by sketching a graph, as shown in Figure 11.14.

x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(x)$	0.5132	0.5013	0.5001	?	0.4999	0.4988	0.4881

 **CHECKPOINT** Now try Exercise 17.

The rationalizing technique for evaluating limits is based on multiplication by a convenient form of 1. In Example 3, the convenient form is

$$1 = \frac{\sqrt{x+1} + 1}{\sqrt{x+1} + 1}$$

Prerequisite Skills

To review rationalizing of numerators and denominators, see Appendix G, Study Capsule 1.

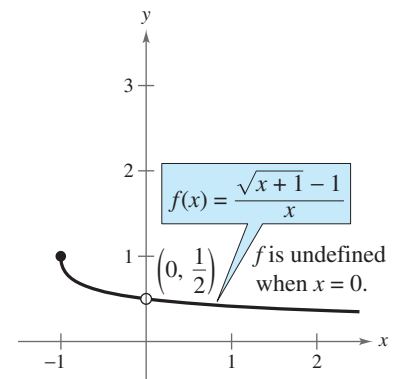


Figure 11.14

Using Technology

The dividing out and rationalizing techniques may not work well for finding limits of nonalgebraic functions. You often need to use more sophisticated analytic techniques to find limits of these types of functions.

Example 4 Approximating a Limit

Approximate the limit: $\lim_{x \rightarrow 0} (1 + x)^{1/x}$.

Numerical Solution

Let $f(x) = (1 + x)^{1/x}$. Because you are finding the limit when $x = 0$, use the *table* feature of a graphing utility to create a table that shows the values of f for x starting at $x = -0.01$ and setting the table step to 0.001, as shown in Figure 11.15. Because 0 is halfway between -0.001 and 0.001 , use the average of the values of f at these two x -coordinates to estimate the limit as follows.

$$\lim_{x \rightarrow 0} (1 + x)^{1/x} \approx \frac{2.7196 + 2.7169}{2} = 2.71825$$

The actual limit can be found algebraically to be $e \approx 2.71828$.

X	Y1
-0.003	2.72224
-0.002	2.721
-0.001	2.7196
0.001	ERR
0.002	2.7169
0.003	2.7142

Figure 11.15

CHECKPOINT Now try Exercise 37.

Graphical Solution

To approximate the limit graphically, graph the function $y = (1 + x)^{1/x}$, as shown in Figure 11.16. Using the *zoom* and *trace* features of the graphing utility, choose two points on the graph of f , such as

$$(-0.00017, 2.7185) \quad \text{and} \quad (0.00017, 2.7181)$$

as shown in Figure 11.17. Because the x -coordinates of these two points are equidistant from 0, you can approximate the limit to be the average of the y -coordinates. That is,

$$\lim_{x \rightarrow 0} (1 + x)^{1/x} \approx \frac{2.7185 + 2.7181}{2} = 2.7183.$$

The actual limit can be found algebraically to be $e \approx 2.71828$.

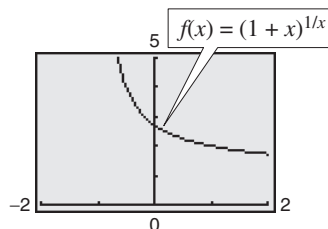


Figure 11.16

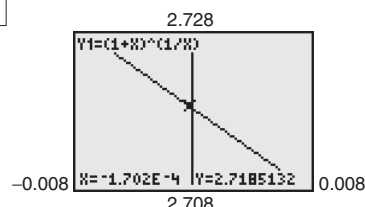


Figure 11.17

Example 5 Approximating a Limit Graphically

Approximate the limit: $\lim_{x \rightarrow 0} \frac{\sin x}{x}$.

Solution

Direct substitution produces the indeterminate form $\frac{0}{0}$. To approximate the limit, begin by using a graphing utility to graph $f(x) = (\sin x)/x$, as shown in Figure 11.18. Then use the *zoom* and *trace* features of the graphing utility to choose a point on each side of 0, such as $(-0.0012467, 0.9999997)$ and $(0.0012467, 0.9999997)$. Finally, approximate the limit as the average of the y -coordinates of these two points, $\lim_{x \rightarrow 0} (\sin x)/x \approx 0.9999997$. It can be shown algebraically that this limit is exactly 1.

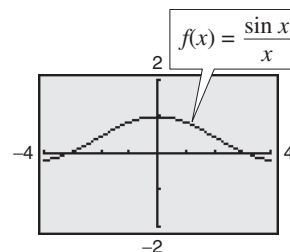


Figure 11.18

CHECKPOINT Now try Exercise 41.

TECHNOLOGY TIP The graphs shown in Figures 11.16 and 11.18 appear to be continuous at $x = 0$. But when you try to use the *trace* or the *value* feature of a graphing utility to determine the value of y when $x = 0$, there is no value given. Some graphing utilities can show breaks or holes in a graph when an appropriate viewing window is used. Because the holes in the graphs in Figures 11.16 and 11.18 occur on the y -axis, the holes are not visible.

TECHNOLOGY SUPPORT

For instructions on how to use the *zoom* and *trace* features and the *value* feature, see Appendix A; for specific keystrokes, go to this textbook's *Online Study Center*.

One-Sided Limits

In Section 11.1, you saw that one way in which a limit can fail to exist is when a function approaches a different value from the left side of c than it approaches from the right side of c . This type of behavior can be described more concisely with the concept of a **one-sided limit**.

$$\lim_{x \rightarrow c^-} f(x) = L_1 \text{ or } f(x) \rightarrow L_1 \text{ as } x \rightarrow c^- \quad \text{Limit from the left}$$

$$\lim_{x \rightarrow c^+} f(x) = L_2 \text{ or } f(x) \rightarrow L_2 \text{ as } x \rightarrow c^+ \quad \text{Limit from the right}$$

Example 6 Evaluating One-Sided Limits

Find the limit as $x \rightarrow 0$ from the left and the limit as $x \rightarrow 0$ from the right for

$$f(x) = \frac{|2x|}{x}.$$

Solution

From the graph of f , shown in Figure 11.19, you can see that $f(x) = -2$ for all $x < 0$. Therefore, the limit from the left is

$$\lim_{x \rightarrow 0^-} \frac{|2x|}{x} = -2. \quad \text{Limit from the left}$$

Because $f(x) = 2$ for all $x > 0$, the limit from the right is

$$\lim_{x \rightarrow 0^+} \frac{|2x|}{x} = 2. \quad \text{Limit from the right}$$

CHECKPOINT Now try Exercise 53.

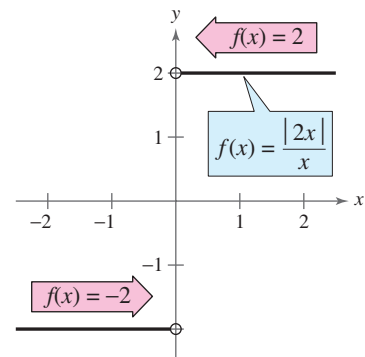


Figure 11.19

In Example 6, note that the function approaches different limits from the left and from the right. In such cases, the limit of $f(x)$ as $x \rightarrow c$ does not exist. For the limit of a function to exist as $x \rightarrow c$, it must be true that both one-sided limits exist and are equal.

Existence of a Limit

If f is a function and c and L are real numbers, then

$$\lim_{x \rightarrow c} f(x) = L$$

if and only if both the left and right limits *exist* and are *equal* to L .

Example 7 Finding One-Sided Limits

Find the limit of $f(x)$ as x approaches 1.

$$f(x) = \begin{cases} 4 - x, & x < 1 \\ 4x - x^2, & x > 1 \end{cases}$$

Solution

Remember that you are concerned about the value of f near $x = 1$ rather than at $x = 1$. So, for $x < 1$, $f(x)$ is given by $4 - x$, and you can use direct substitution to obtain

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} (4 - x) = 4 - 1 = 3.$$

For $x > 1$, $f(x)$ is given by $4x - x^2$, and you can use direct substitution to obtain

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} (4x - x^2) = 4(1) - 1^2 = 3.$$

Because the one-sided limits both exist and are equal to 3, it follows that

$$\lim_{x \rightarrow 1} f(x) = 3.$$

The graph in Figure 11.20 confirms this conclusion.

Prerequisite Skills

For a review of piecewise-defined functions, see Section 1.2.

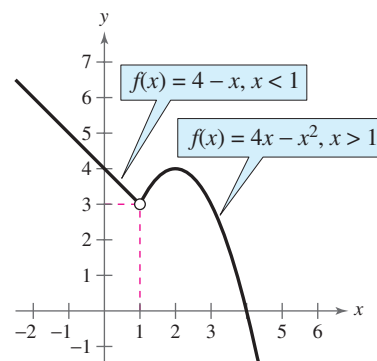


Figure 11.20

CHECKPOINT Now try Exercise 57.

Example 8 Comparing Limits from the Left and Right

To ship a package overnight, a delivery service charges \$17.80 for the first pound and \$1.40 for each additional pound or portion of a pound. Let x represent the weight of a package and let $f(x)$ represent the shipping cost. Show that the limit of $f(x)$ as $x \rightarrow 2$ does not exist.

$$f(x) = \begin{cases} 17.80, & 0 < x \leq 1 \\ 19.20, & 1 < x \leq 2 \\ 20.60, & 2 < x \leq 3 \end{cases}$$

Solution

The graph of f is shown in Figure 11.21. The limit of $f(x)$ as x approaches 2 from the left is

$$\lim_{x \rightarrow 2^-} f(x) = 19.20$$

whereas the limit of $f(x)$ as x approaches 2 from the right is

$$\lim_{x \rightarrow 2^+} f(x) = 20.60.$$

Because these one-sided limits are not equal, the limit of $f(x)$ as $x \rightarrow 2$ does not exist.

CHECKPOINT Now try Exercise 81.

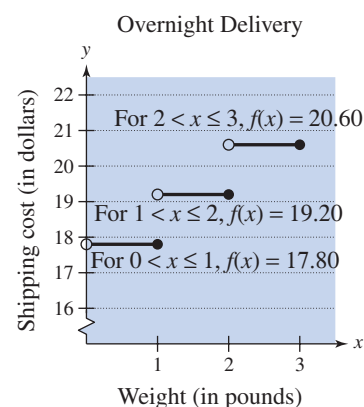


Figure 11.21

A Limit from Calculus

In the next section, you will study an important type of limit from calculus—the limit of a *difference quotient*.

Example 9 Evaluating a Limit from Calculus



For the function given by $f(x) = x^2 - 1$, find

$$\lim_{h \rightarrow 0} \frac{f(3 + h) - f(3)}{h}.$$

Solution

Direct substitution produces an indeterminate form.

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{f(3 + h) - f(3)}{h} &= \lim_{h \rightarrow 0} \frac{[(3 + h)^2 - 1] - [(3)^2 - 1]}{h} \\ &= \lim_{h \rightarrow 0} \frac{9 + 6h + h^2 - 1 - 9 + 1}{h} \\ &= \lim_{h \rightarrow 0} \frac{6h + h^2}{h} \\ &= \frac{0}{0} \end{aligned}$$

By factoring and dividing out, you obtain the following.

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{f(3 + h) - f(3)}{h} &= \lim_{h \rightarrow 0} \frac{6h + h^2}{h} = \lim_{h \rightarrow 0} \frac{h(6 + h)}{h} \\ &= \lim_{h \rightarrow 0} (6 + h) \\ &= 6 + 0 \\ &= 6 \end{aligned}$$

So, the limit is 6.



CHECKPOINT Now try Exercise 73.

Note that for any x -value, the limit of a difference quotient is an expression of the form

$$\lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h}.$$

Direct substitution into the difference quotient always produces the indeterminate form $\frac{0}{0}$. For instance,

$$\lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h} = \frac{f(x + 0) - f(x)}{0} = \frac{f(x) - f(x)}{0} = \frac{0}{0}.$$

Prerequisite Skills

For a review of evaluating difference quotients, refer to Section 1.2.

11.2 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

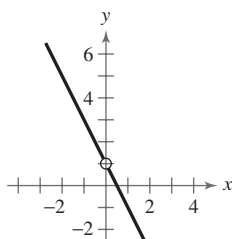
Vocabulary Check

Fill in the blanks.

- To evaluate the limit of a rational function that has common factors in its numerator and denominator, use the _____.
- The fraction $\frac{0}{0}$ has no meaning as a real number and therefore is called an _____.
- The limit $\lim_{x \rightarrow c} f(x) = L$ is an example of a _____.
- The limit of a _____ is an expression of the form $\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$.

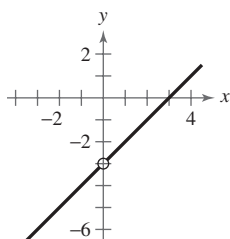
In Exercises 1–4, use the graph to determine each limit (if it exists). Then identify another function that agrees with the given function at all but one point.

1. $g(x) = \frac{-2x^2 + x}{x}$



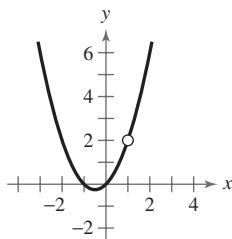
- $\lim_{x \rightarrow 0} g(x)$
- $\lim_{x \rightarrow -1} g(x)$
- $\lim_{x \rightarrow -2} g(x)$

2. $h(x) = \frac{x^2 - 3x}{x}$



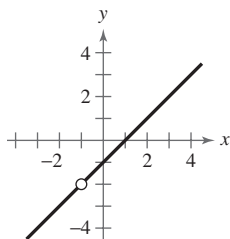
- $\lim_{x \rightarrow -2} h(x)$
- $\lim_{x \rightarrow 0} h(x)$
- $\lim_{x \rightarrow 3} h(x)$

3. $g(x) = \frac{x^3 - x}{x - 1}$



- $\lim_{x \rightarrow 1} g(x)$
- $\lim_{x \rightarrow -1} g(x)$
- $\lim_{x \rightarrow 0} g(x)$

4. $f(x) = \frac{x^2 - 1}{x + 1}$



- $\lim_{x \rightarrow 1} f(x)$
- $\lim_{x \rightarrow 2} f(x)$
- $\lim_{x \rightarrow -1} f(x)$

In Exercises 5–28, find the limit (if it exists). Use a graphing utility to verify your result graphically.

- $\lim_{x \rightarrow 6} \frac{x - 6}{x^2 - 36}$
- $\lim_{x \rightarrow 9} \frac{9 - x}{x^2 - 81}$
- $\lim_{x \rightarrow -1} \frac{1 - 2x - 3x^2}{1 + x}$
- $\lim_{x \rightarrow -4} \frac{2x^2 + 7x - 4}{x + 4}$
- $\lim_{t \rightarrow 2} \frac{t^3 - 8}{t - 2}$
- $\lim_{a \rightarrow -4} \frac{a^3 + 64}{a + 4}$
- $\lim_{x \rightarrow 1} \frac{x^4 - 1}{x^4 - 3x^2 - 4}$
- $\lim_{x \rightarrow 2} \frac{x^4 - 2x^2 - 8}{x^4 - 6x^2 + 8}$
- $\lim_{x \rightarrow -1} \frac{x^3 + 2x^2 - x - 2}{x^3 + 4x^2 - x - 4}$
- $\lim_{x \rightarrow -3} \frac{x^3 + 2x^2 - 9x - 18}{x^3 + x^2 - 9x - 9}$
- $\lim_{x \rightarrow 2} \frac{x^3 + 2x^2 - 5x - 6}{x^3 - 7x + 6}$
- $\lim_{x \rightarrow 3} \frac{x^3 - 4x^2 - 3x + 18}{x^3 - 4x^2 + x + 6}$
- $\lim_{y \rightarrow 0} \frac{\sqrt{5 + y} - \sqrt{5}}{y}$
- $\lim_{z \rightarrow 0} \frac{\sqrt{7 - z} - \sqrt{7}}{z}$
- $\lim_{x \rightarrow -3} \frac{\sqrt{x + 7} - 2}{x + 3}$
- $\lim_{x \rightarrow 2} \frac{4 - \sqrt{18 - x}}{x - 2}$
- $\lim_{x \rightarrow 0} \frac{\frac{1}{x + 1} - 1}{x}$
- $\lim_{x \rightarrow 0} \frac{\frac{1}{x - 8} + \frac{1}{8}}{x}$
- $\lim_{x \rightarrow 0} \frac{\sec x}{\tan x}$
- $\lim_{x \rightarrow \pi/2} \frac{1 - \sin x}{\cos x}$
- $\lim_{x \rightarrow 0} \frac{\cos 2x}{\cot 2x}$
- $\lim_{x \rightarrow 0} \frac{\sin x - x}{x}$
- $\lim_{x \rightarrow 0} \frac{1 + \cos x}{x}$
- $\lim_{x \rightarrow 0} \frac{\sqrt{x + 3} - \sqrt{3}}{x}$
- $\lim_{x \rightarrow 0} \frac{\sqrt{x + 4} - 2}{x}$

In Exercises 29–36, use a graphing utility to graph the function and approximate the limit.

- $\lim_{x \rightarrow 0} \frac{\sqrt{x + 3} - \sqrt{3}}{x}$
- $\lim_{x \rightarrow 0} \frac{\sqrt{x + 4} - 2}{x}$

31. $\lim_{x \rightarrow 0} \frac{\sqrt{2x+1} - 1}{x}$

32. $\lim_{x \rightarrow 9} \frac{3 - \sqrt{x}}{x - 9}$

33. $\lim_{x \rightarrow 2} \frac{x^5 - 32}{x - 2}$

34. $\lim_{x \rightarrow 1} \frac{x^4 - 1}{x - 1}$

35. $\lim_{x \rightarrow 0} \frac{\frac{1}{x+4} - \frac{1}{4}}{x}$

36. $\lim_{x \rightarrow 0} \frac{\frac{1}{2+x} - \frac{1}{2}}{x}$

In Exercises 37–48, use a graphing utility to graph the function and approximate the limit. Write an approximation that is accurate to three decimal places.

37. $\lim_{x \rightarrow 0} \frac{e^{2x} - 1}{x}$

38. $\lim_{x \rightarrow 0} \frac{1 - e^{-x}}{x}$

39. $\lim_{x \rightarrow 0^+} (x \ln x)$

40. $\lim_{x \rightarrow 0^+} (x^2 \ln x)$

41. $\lim_{x \rightarrow 0} \frac{\sin 2x}{x}$

42. $\lim_{x \rightarrow 0} \frac{\sin 3x}{x}$

43. $\lim_{x \rightarrow 0} \frac{\tan x}{x}$

44. $\lim_{x \rightarrow 0} \frac{1 - \cos 2x}{x}$

45. $\lim_{x \rightarrow 1} \frac{1 - \sqrt[3]{x}}{1 - x}$

46. $\lim_{x \rightarrow 1} \frac{\sqrt[3]{x} - x}{x - 1}$

47. $\lim_{x \rightarrow 0} (1 - x)^{2/x}$

48. $\lim_{x \rightarrow 0} (1 + 2x)^{1/x}$

Graphical, Numerical, and Algebraic Analysis In Exercises 49–52, (a) graphically approximate the limit (if it exists) by using a graphing utility to graph the function, (b) numerically approximate the limit (if it exists) by using the *table* feature of a graphing utility to create a table, and (c) algebraically evaluate the limit (if it exists) by the appropriate technique(s).

49. $\lim_{x \rightarrow 1^-} \frac{x - 1}{x^2 - 1}$

50. $\lim_{x \rightarrow 5^+} \frac{5 - x}{25 - x^2}$

51. $\lim_{x \rightarrow 16^+} \frac{4 - \sqrt{x}}{x - 16}$

52. $\lim_{x \rightarrow 0^-} \frac{\sqrt{x+2} - \sqrt{2}}{x}$

In Exercises 53–60, graph the function. Determine the limit (if it exists) by evaluating the corresponding one-sided limits.

53. $\lim_{x \rightarrow 6} \frac{|x - 6|}{x - 6}$

54. $\lim_{x \rightarrow 2} \frac{|x - 2|}{x - 2}$

55. $\lim_{x \rightarrow 1} \frac{1}{x^2 + 1}$

56. $\lim_{x \rightarrow 1} \frac{1}{x^2 - 1}$

57. $\lim_{x \rightarrow 2} f(x)$ where $f(x) = \begin{cases} x - 1, & x \leq 2 \\ 2x - 3, & x > 2 \end{cases}$

58. $\lim_{x \rightarrow 1} f(x)$ where $f(x) = \begin{cases} 2x + 1, & x < 1 \\ 4 - x^2, & x \geq 1 \end{cases}$

59. $\lim_{x \rightarrow 1} f(x)$ where $f(x) = \begin{cases} 4 - x^2, & x \leq 1 \\ 3 - x, & x > 1 \end{cases}$

60. $\lim_{x \rightarrow 0} f(x)$ where $f(x) = \begin{cases} 4 - x^2, & x \leq 0 \\ x + 4, & x > 0 \end{cases}$

In Exercises 61–66, use a graphing utility to graph the function and the equations $y = x$ and $y = -x$ in the same viewing window. Use the graph to find $\lim_{x \rightarrow 0} f(x)$.

61. $f(x) = x \cos x$

62. $f(x) = |x \sin x|$

63. $f(x) = |x| \sin x$

64. $f(x) = |x| \cos x$

65. $f(x) = x \sin \frac{1}{x}$

66. $f(x) = x \cos \frac{1}{x}$

In Exercises 67 and 68, state which limit can be evaluated by using direct substitution. Then evaluate or approximate each limit.

67. (a) $\lim_{x \rightarrow 0} x^2 \sin x^2$

(b) $\lim_{x \rightarrow 0} \frac{\sin x^2}{x^2}$

68. (a) $\lim_{x \rightarrow 0} \frac{x}{\cos x}$

(b) $\lim_{x \rightarrow 0} \frac{1 - \cos x}{x}$

f In Exercises 69–76, find $\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$.

69. $f(x) = 3x - 1$

70. $f(x) = 5 - 6x$

71. $f(x) = \sqrt{x}$

72. $f(x) = \sqrt{x - 2}$

73. $f(x) = x^2 - 3x$

74. $f(x) = 4 - 2x - x^2$

75. $f(x) = \frac{1}{x + 2}$

76. $f(x) = \frac{1}{x - 1}$

Free-Falling Object In Exercises 77 and 78, use the position function $s(t) = -16t^2 + 128$, which gives the height (in feet) of a free-falling object. The velocity at time $t = a$ seconds is given by

$\lim_{t \rightarrow a} \frac{s(a) - s(t)}{a - t}$.

77. Find the velocity when $t = 1$ second.

78. Find the velocity when $t = 2$ seconds.

79. Communications The cost of a cellular phone call within your calling area is \$1.00 for the first minute and \$0.25 for each additional minute or portion of a minute. A model for the cost C is given by $C(t) = 1.00 - 0.25\lceil -(t - 1) \rceil$, where t is the time in minutes. (Recall from Section 1.3 that $f(x) = \lceil x \rceil$ = the greatest integer less than or equal to x .)

(a) Sketch the graph of C for $0 < t \leq 5$.

(b) Complete the table and observe the behavior of C as t approaches 3.5. Use the graph from part (a) and the table to find $\lim_{t \rightarrow 3.5} C(t)$.

t	3	3.3	3.4	3.5	3.6	3.7	4
C				?			

- (c) Complete the table and observe the behavior of C as t approaches 3. Does the limit of $C(t)$ as t approaches 3 exist? Explain.

t	2	2.5	2.9	3	3.1	3.5	4
C				?			

- 80. Communications** The cost of a cellular phone call within your calling area is \$1.25 for the first minute and \$0.15 for each additional minute or portion of a minute. A model for the cost C is given by $C(t) = 1.25 - 0.15\lceil -(t - 1) \rceil$, where t is the time in minutes. (Recall from Section 1.3 that $f(x) = \lceil x \rceil$ = the greatest integer less than or equal to x .)

- (a) Sketch the graph of C for $0 < t \leq 5$.
 (b) Complete the table and observe the behavior of C as t approaches 3.5. Use the graph from part (a) and the table to find $\lim_{t \rightarrow 3.5} C(t)$.

t	3	3.3	3.4	3.5	3.6	3.7	4
C				?			

- (c) Complete the table and observe the behavior of C as t approaches 3. Does the limit of $C(t)$ as t approaches 3 exist? Explain.

t	2	2.5	2.9	3	3.1	3.5	4
C				?			

- 81. Salary Contract** A union contract guarantees a 20% salary increase yearly for 3 years. For a current salary of \$32,500, the salary $f(t)$ (in thousands of dollars) for the next 3 years is given by

$$f(t) = \begin{cases} 32.50, & 0 < t \leq 1 \\ 39.00, & 1 < t \leq 2 \\ 46.80, & 2 < t \leq 3 \end{cases}$$

where t represents the time in years. Show that the limit of f as $t \rightarrow 2$ does not exist.

- 82. Consumer Awareness** The cost of sending a package overnight is \$14.40 for the first pound and \$3.90 for each additional pound or portion of a pound. A plastic mailing bag can hold up to 3 pounds. The cost $f(x)$ of sending a package in a plastic mailing bag is given by

$$f(x) = \begin{cases} 14.40, & 0 < x \leq 1 \\ 18.30, & 1 < x \leq 2 \\ 22.20, & 2 < x \leq 3 \end{cases}$$

where x represents the weight of the package (in pounds). Show that the limit of f as $x \rightarrow 1$ does not exist.

Synthesis

True or False? In Exercises 83 and 84, determine whether the statement is true or false. Justify your answer.

- 83.** When your attempt to find the limit of a rational function yields the indeterminate form $\frac{0}{0}$, the rational function's numerator and denominator have a common factor.
84. If $f(c) = L$, then $\lim_{x \rightarrow c} f(x) = L$.

85. Think About It

- (a) Sketch the graph of a function for which $f(2)$ is defined but for which the limit of $f(x)$ as x approaches 2 does not exist.
 (b) Sketch the graph of a function for which the limit of $f(x)$ as x approaches 1 is 4 but for which $f(1) \neq 4$.

- 86. Writing** Consider the limit of the rational function $p(x)/q(x)$. What conclusion can you make if direct substitution produces each expression? Write a short paragraph explaining your reasoning.

(a) $\lim_{x \rightarrow c} \frac{p(x)}{q(x)} = \frac{0}{1}$ (b) $\lim_{x \rightarrow c} \frac{p(x)}{q(x)} = \frac{1}{1}$
 (c) $\lim_{x \rightarrow c} \frac{p(x)}{q(x)} = \frac{1}{0}$ (d) $\lim_{x \rightarrow c} \frac{p(x)}{q(x)} = \frac{0}{0}$

Skills Review

- 87.** Write an equation of the line that passes through $(6, -10)$ and is perpendicular to the line that passes through $(4, -6)$ and $(3, -4)$.
88. Write an equation of the line that passes through $(1, -1)$ and is parallel to the line that passes through $(3, -3)$ and $(5, -2)$.

In Exercises 89–94, identify the type of conic algebraically. Then use a graphing utility to graph the conic.

89. $r = \frac{3}{1 + \cos \theta}$ **90.** $r = \frac{12}{3 + 2 \sin \theta}$
91. $r = \frac{9}{2 + 3 \cos \theta}$ **92.** $r = \frac{4}{4 + \cos \theta}$
93. $r = \frac{5}{1 - \sin \theta}$ **94.** $r = \frac{6}{3 - 4 \sin \theta}$

In Exercises 95–98, determine whether the vectors are orthogonal, parallel, or neither.

- 95.** $\langle 7, -2, 3 \rangle, \langle -1, 4, 5 \rangle$
96. $\langle 5, 5, 0 \rangle, \langle 0, 5, 1 \rangle$
97. $\langle -4, 3, -6 \rangle, \langle 12, -9, 18 \rangle$
98. $\langle 2, -3, 1 \rangle, \langle -2, 2, 2 \rangle$

11.3 The Tangent Line Problem

Tangent Line to a Graph

Calculus is a branch of mathematics that studies rates of change of functions. If you go on to take a course in calculus, you will learn that rates of change have many applications in real life.

Earlier in the text, you learned how the slope of a line indicates the rate at which a line rises or falls. For a line, this rate (or slope) is the same at every point on the line. For graphs other than lines, the rate at which the graph rises or falls changes from point to point. For instance, in Figure 11.22, the parabola is rising more quickly at the point (x_1, y_1) than it is at the point (x_2, y_2) . At the vertex (x_3, y_3) , the graph levels off, and at the point (x_4, y_4) , the graph is falling.

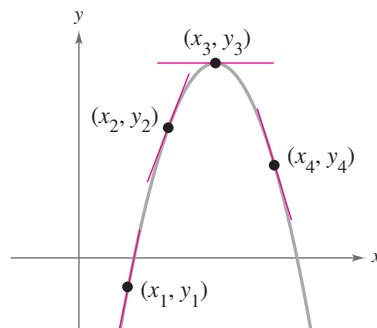


Figure 11.22

To determine the rate at which a graph rises or falls at a *single point*, you can find the slope of the tangent line at that point. In simple terms, the **tangent line** to the graph of a function f at a point $P(x_1, y_1)$ is the line that best approximates the slope of the graph at the point. Figure 11.23 shows other examples of tangent lines.

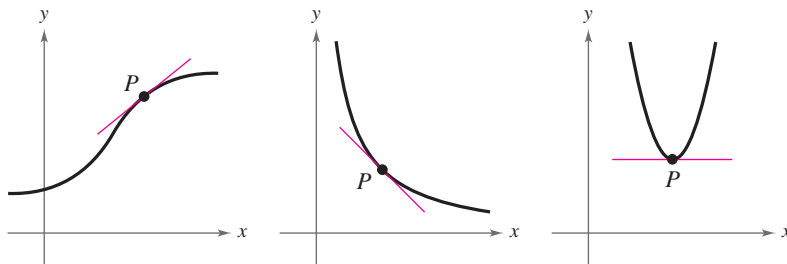


Figure 11.23

From geometry, you know that a line is tangent to a circle if the line intersects the circle at only one point. Tangent lines to noncircular graphs, however, can intersect the graph at more than one point. For instance, in the first graph in Figure 11.23, if the tangent line were extended, it would intersect the graph at a point other than the point of tangency.

What you should learn

- Use a tangent line to approximate the slope of a graph at a point.
- Use the limit definition of slope to find exact slopes of graphs.
- Find derivatives of functions and use derivatives to find slopes of graphs.

Why you should learn it

The derivative, or the slope of the tangent line to the graph of a function at a point, can be used to analyze rates of change. For instance, in Exercise 65 on page 809, the derivative is used to analyze the rate of change of the volume of a spherical balloon.



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Prerequisite Skills

For a review of the slopes of lines, see Section 1.1.

Slope of a Graph

Because a tangent line approximates the slope of a graph at a point, the problem of finding the slope of a graph at a point is the same as finding the slope of the tangent line at the point.

Example 1 Visually Approximating the Slope of a Graph

Use the graph in Figure 11.24 to approximate the slope of the graph of $f(x) = x^2$ at the point $(1, 1)$.

Solution

From the graph of $f(x) = x^2$, you can see that the tangent line at $(1, 1)$ rises approximately two units for each unit change in x . So, you can estimate the slope of the tangent line at $(1, 1)$ to be

$$\begin{aligned}\text{Slope} &= \frac{\text{change in } y}{\text{change in } x} \\ &\approx \frac{2}{1} \\ &= 2.\end{aligned}$$

Because the tangent line at the point $(1, 1)$ has a slope of about 2, you can conclude that the graph of f has a slope of about 2 at the point $(1, 1)$.

CHECKPOINT Now try Exercise 1.

When you are visually approximating the slope of a graph, remember that the scales on the horizontal and vertical axes may differ. When this happens (as it frequently does in applications), the slope of the tangent line is distorted, and you must be careful to account for the difference in scales.

Example 2 Approximating the Slope of a Graph



Figure 11.25 graphically depicts the monthly normal temperatures (in degrees Fahrenheit) for Dallas, Texas. Approximate the slope of this graph at the indicated point and give a physical interpretation of the result. (Source: National Climatic Data Center)

Solution

From the graph, you can see that the tangent line at the given point falls approximately 16 units for each two-unit change in x . So, you can estimate the slope at the given point to be

$$\text{Slope} = \frac{\text{change in } y}{\text{change in } x} \approx \frac{-16}{2} = -8 \text{ degrees per month.}$$

This means that you can expect the monthly normal temperature in November to be about 8 degrees lower than the normal temperature in October.

CHECKPOINT Now try Exercise 3.

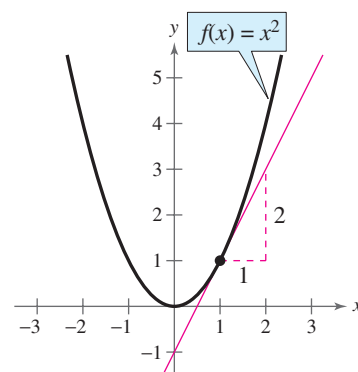


Figure 11.24

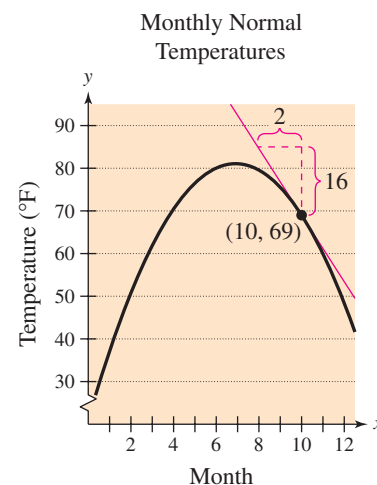


Figure 11.25

Slope and the Limit Process

In Examples 1 and 2, you approximated the slope of a graph at a point by creating a graph and then “eyeballing” the tangent line at the point of tangency. A more systematic method of approximating tangent lines makes use of a **secant line** through the point of tangency and a second point on the graph, as shown in Figure 11.26. If $(x, f(x))$ is the point of tangency and $(x + h, f(x + h))$ is a second point on the graph of f , the slope of the secant line through the two points is given by

$$m_{\text{sec}} = \frac{f(x + h) - f(x)}{h} \quad \text{Slope of secant line}$$

The right side of this equation is called the **difference quotient**. The denominator h is the *change in x* , and the numerator is the *change in y* . The beauty of this procedure is that you obtain more and more accurate approximations of the slope of the tangent line by choosing points closer and closer to the point of tangency, as shown in Figure 11.27.

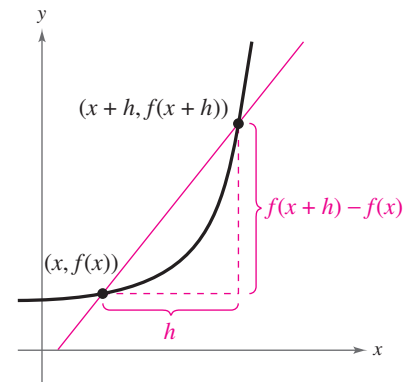
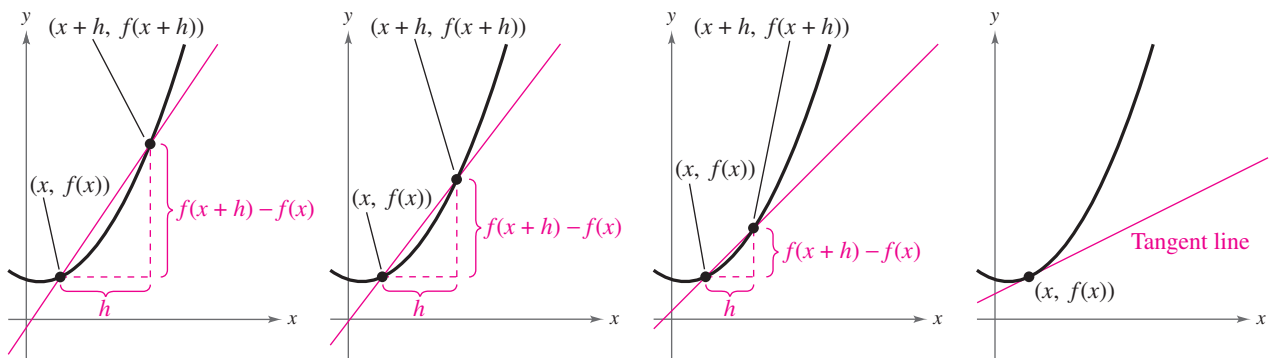


Figure 11.26



As h approaches 0, the secant line approaches the tangent line.

Figure 11.27

Using the limit process, you can find the *exact* slope of the tangent line at $(x, f(x))$.

Definition of the Slope of a Graph

The **slope** m of the graph of f at the point $(x, f(x))$ is equal to the slope of its tangent line at $(x, f(x))$, and is given by

$$\begin{aligned} m &= \lim_{h \rightarrow 0} m_{\text{sec}} \\ &= \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h} \end{aligned}$$

provided this limit exists.

From the above definition and from Section 11.2, you can see that the difference quotient is used frequently in calculus. Using the difference quotient to find the slope of a tangent line to a graph is a major concept of calculus.

Example 3 Finding the Slope of a Graph

Find the slope of the graph of $f(x) = x^2$ at the point $(-2, 4)$.

Solution

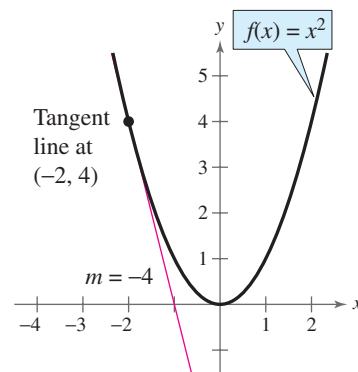
Find an expression that represents the slope of a secant line at $(-2, 4)$.

$$\begin{aligned} m_{\text{sec}} &= \frac{f(-2+h) - f(-2)}{h} && \text{Set up difference quotient.} \\ &= \frac{(-2+h)^2 - (-2)^2}{h} && \text{Substitute into } f(x) = x^2. \\ &= \frac{4 - 4h + h^2 - 4}{h} && \text{Expand terms.} \\ &= \frac{-4h + h^2}{h} && \text{Simplify.} \\ &= \frac{h(-4 + h)}{h} && \text{Factor and divide out.} \\ &= -4 + h, \quad h \neq 0 && \text{Simplify.} \end{aligned}$$

Next, take the limit of m_{sec} as h approaches 0.

$$m = \lim_{h \rightarrow 0} m_{\text{sec}} = \lim_{h \rightarrow 0} (-4 + h) = -4 + 0 = -4$$

The graph has a slope of -4 at the point $(-2, 4)$, as shown in Figure 11.28.

**Figure 11.28**

CHECKPOINT Now try Exercise 5.

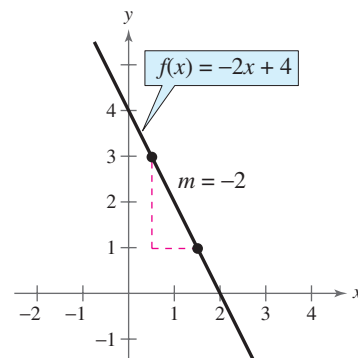
Example 4 Finding the Slope of a Graph

Find the slope of $f(x) = -2x + 4$.

Solution

$$\begin{aligned} m &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} && \text{Set up difference quotient.} \\ &= \lim_{h \rightarrow 0} \frac{[-2(x+h) + 4] - (-2x + 4)}{h} && \text{Substitute into } f(x) = -2x + 4. \\ &= \lim_{h \rightarrow 0} \frac{-2x - 2h + 4 + 2x - 4}{h} && \text{Expand terms.} \\ &= \lim_{h \rightarrow 0} \frac{-2h}{h} && \text{Divide out.} \\ &= -2 && \text{Simplify.} \end{aligned}$$

You know from your study of linear functions that the line given by $f(x) = -2x + 4$ has a slope of -2 , as shown in Figure 11.29. This conclusion is consistent with that obtained by the limit definition of slope, as shown above.

**Figure 11.29**

CHECKPOINT Now try Exercise 7.

It is important that you see the difference between the ways the difference quotients were set up in Examples 3 and 4. In Example 3, you were finding the slope of a graph at a specific point $(c, f(c))$. To find the slope in such a case, you can use the following form of the difference quotient.

$$m = \lim_{h \rightarrow 0} \frac{f(c+h) - f(c)}{h} \quad \text{Slope at specific point}$$

In Example 4, however, you were finding a *formula* for the slope at *any* point on the graph. In such cases, you should use x , rather than c , in the difference quotient.

$$m = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \quad \text{Formula for slope}$$

Except for linear functions, this form will always produce a function of x , which can then be evaluated to find the slope at any desired point.

Example 5 Finding a Formula for the Slope of a Graph

Find a formula for the slope of the graph of $f(x) = x^2 + 1$. What are the slopes at the points $(-1, 2)$ and $(2, 5)$?

Solution

$$m_{\text{sec}} = \frac{f(x+h) - f(x)}{h} \quad \text{Set up difference quotient.}$$

$$= \frac{[(x+h)^2 + 1] - (x^2 + 1)}{h} \quad \text{Substitute into } f(x) = x^2 + 1.$$

$$= \frac{x^2 + 2xh + h^2 + 1 - x^2 - 1}{h} \quad \text{Expand terms.}$$

$$= \frac{2xh + h^2}{h} \quad \text{Simplify.}$$

$$= \frac{h(2x + h)}{h} \quad \text{Factor and divide out.}$$

$$= 2x + h, \quad h \neq 0 \quad \text{Simplify.}$$

Next, take the limit of m_{sec} as h approaches 0.

$$m = \lim_{h \rightarrow 0} m_{\text{sec}} = \lim_{h \rightarrow 0} (2x + h) = 2x + 0 = 2x$$

Using the formula $m = 2x$ for the slope at $(x, f(x))$, you can find the slope at the specified points. At $(-1, 2)$, the slope is

$$m = 2(-1) = -2$$

and at $(2, 5)$, the slope is

$$m = 2(2) = 4.$$

The graph of f is shown in Figure 11.30.

 **CHECKPOINT** Now try Exercise 13.

TECHNOLOGY TIP

Try verifying the result in Example 5 by using a graphing utility to graph the function and the tangent lines at $(-1, 2)$ and $(2, 5)$ as

$$y_1 = x^2 + 1$$

$$y_2 = -2x$$

$$y_3 = 4x - 3$$

in the same viewing window. Some graphing utilities even have a *tangent* feature that automatically graphs the tangent line to a curve at a given point. If you have such a graphing utility, try verifying the solution of Example 5 using this feature. For instructions on how to use the *tangent* feature, see Appendix A; for specific keystrokes, go to this textbook's *Online Study Center*.

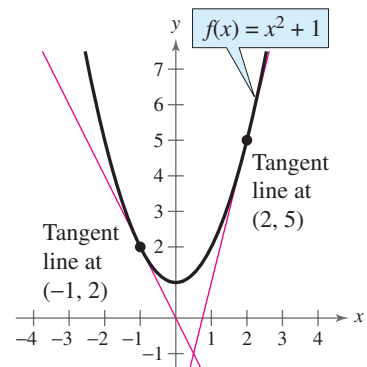


Figure 11.30

The Derivative of a Function

In Example 5, you started with the function $f(x) = x^2 + 1$ and used the limit process to derive another function, $m = 2x$, that represents the slope of the graph of f at the point $(x, f(x))$. This derived function is called the **derivative** of f at x . It is denoted by $f'(x)$, which is read as “ f prime of x .”

Definition of the Derivative

The **derivative** of f at x is given by

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

provided this limit exists.

Remember that the derivative $f'(x)$ is a formula for the slope of the tangent line to the graph of f at the point $(x, f(x))$.

Example 6 Finding a Derivative



Find the derivative of $f(x) = 3x^2 - 2x$.

Solution

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{[3(x+h)^2 - 2(x+h)] - (3x^2 - 2x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{3x^2 + 6xh + 3h^2 - 2x - 2h - 3x^2 + 2x}{h} \\ &= \lim_{h \rightarrow 0} \frac{6xh + 3h^2 - 2h}{h} \\ &= \lim_{h \rightarrow 0} \frac{h(6x + 3h - 2)}{h} \\ &= \lim_{h \rightarrow 0} (6x + 3h - 2) \\ &= 6x + 3(0) - 2 \\ &= 6x - 2 \end{aligned}$$

So, the derivative of $f(x) = 3x^2 - 2x$ is $f'(x) = 6x - 2$.

CHECKPOINT Now try Exercise 29.

Note that in addition to $f'(x)$, other notations can be used to denote the derivative of $y = f(x)$. The most common are

$$\frac{dy}{dx}, \quad y', \quad \frac{d}{dx}[f(x)], \quad \text{and} \quad D_x[y].$$

STUDY TIP

In Section 1.1, you studied the slope of a line, which represents the *average rate of change* over an interval. The derivative of a function is a formula which represents the *instantaneous rate of change* at a point.

Exploration

Use a graphing utility to graph the function $f(x) = 3x^2 - 2x$. Use the *trace* feature to approximate the coordinates of the vertex of this parabola. Then use the derivative of $f(x) = 3x^2 - 2x$ to find the slope of the tangent line at the vertex. Make a conjecture about the slope of the tangent line at the vertex of an arbitrary parabola.

Example 7 Using the Derivative

Find $f'(x)$ for $f(x) = \sqrt{x}$. Then find the slopes of the graph of f at the points $(1, 1)$ and $(4, 2)$ and equations of the tangent lines to the graph at the points.

Solution

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \end{aligned}$$

Because direct substitution yields the indeterminate form $\frac{0}{0}$, you should use the rationalizing technique discussed in Section 11.2 to find the limit.

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \left(\frac{\sqrt{x+h} - \sqrt{x}}{h} \right) \left(\frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \right) \\ &= \lim_{h \rightarrow 0} \frac{(x+h) - x}{h(\sqrt{x+h} + \sqrt{x})} \\ &= \lim_{h \rightarrow 0} \frac{h}{h(\sqrt{x+h} + \sqrt{x})} \\ &= \lim_{h \rightarrow 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} \\ &= \frac{1}{\sqrt{x+0} + \sqrt{x}} = \frac{1}{2\sqrt{x}} \end{aligned}$$

At the point $(1, 1)$, the slope is

$$f'(1) = \frac{1}{2\sqrt{1}} = \frac{1}{2}.$$

An equation of the tangent line at the point $(1, 1)$ is

$$y - y_1 = m(x - x_1)$$

Point-slope form

$$y - 1 = \frac{1}{2}(x - 1)$$

Substitute $\frac{1}{2}$ for m , 1 for x_1 , and 1 for y_1 .

$$y = \frac{1}{2}x + \frac{1}{2}.$$

Tangent line

At the point $(4, 2)$, the slope is

$$f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}.$$

An equation of the tangent line at the point $(4, 2)$ is

$$y - y_1 = m(x - x_1)$$

Point-slope form

$$y - 2 = \frac{1}{4}(x - 4)$$

Substitute $\frac{1}{4}$ for m , 4 for x_1 , and 2 for y_1 .

$$y = \frac{1}{4}x + 1.$$

Tangent line

The graphs of f and the tangent lines at the points $(1, 1)$ and $(4, 2)$ are shown in Figure 11.31.

STUDY TIP

Remember that in order to rationalize the numerator of an expression, you must multiply the numerator and denominator by the conjugate of the numerator.

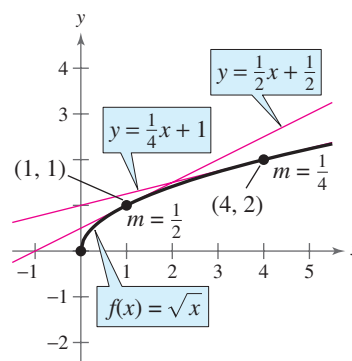


Figure 11.31



Now try Exercise 39.

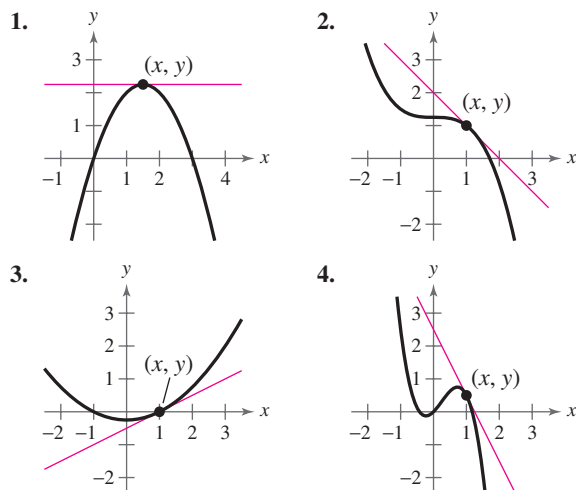
11.3 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Vocabulary Check

Fill in the blanks.

- _____ is the study of the rates of change of functions.
- The _____ to the graph of a function at a point is the line that best approximates the slope of the graph at the point.
- A _____ is a line through the point of tangency and a second point on the graph.
- The slope of the secant line is represented by the _____ $m_{\text{sec}} = \frac{f(x+h) - f(x)}{h}$.
- The _____ of a function f at x represents the slope of the graph of f at the point $(x, f(x))$.

In Exercises 1–4, use the figure to approximate the slope of the curve at the point (x, y) .

In Exercises 5–12, use the limit process to find the slope of the graph of the function at the specified point. Use a graphing utility to confirm your result.

- $g(x) = x^2 - 4x$, $(3, -3)$
- $f(x) = 10x - 2x^2$, $(3, 12)$
- $g(x) = 5 - 2x$, $(1, 3)$
- $h(x) = 2x + 5$, $(-1, -3)$
- $g(x) = \frac{4}{x}$, $(2, 2)$
- $g(x) = \frac{1}{x-2}$, $(4, \frac{1}{2})$
- $h(x) = \sqrt{x}$, $(9, 3)$
- $h(x) = \sqrt{x+10}$, $(-1, 3)$

In Exercises 13–18, find a formula for the slope of the graph of f at the point $(x, f(x))$. Then use it to find the slopes at the two specified points.

- $g(x) = 4 - x^2$
(a) $(0, 4)$
(b) $(-1, 3)$
- $g(x) = x^3$
(a) $(1, 1)$
(b) $(-2, -8)$

- $g(x) = \frac{1}{x+4}$
(a) $(0, \frac{1}{4})$
(b) $(-2, \frac{1}{2})$
- $g(x) = \sqrt{x-1}$
(a) $(5, 2)$
(b) $(10, 3)$
- $f(x) = \frac{1}{x+2}$
(a) $(0, \frac{1}{2})$
(b) $(-1, 1)$
- $f(x) = \sqrt{x-4}$
(a) $(5, 1)$
(b) $(8, 2)$

In Exercises 19–24, use a graphing utility to graph the function and the tangent line at the point $(1, f(1))$. Use the graph to approximate the slope of the tangent line.

- $f(x) = x^2 - 2$
- $f(x) = x^2 - 2x + 1$
- $f(x) = \sqrt{2-x}$
- $f(x) = \sqrt{x+3}$
- $f(x) = \frac{4}{x+1}$
- $f(x) = \frac{3}{2-x}$

In Exercises 25–38, find the derivative of the function.

- $f(x) = 5$
- $f(x) = -1$
- $f(x) = 9 - \frac{1}{3}x$
- $f(x) = -5x + 2$
- $f(x) = 4 - 3x^2$
- $f(x) = x^2 - 3x + 4$
- $f(x) = \frac{1}{x^2}$
- $f(x) = \frac{1}{x^3}$
- $f(x) = \sqrt{x-4}$
- $f(x) = \sqrt{x+8}$
- $f(x) = \frac{1}{x+2}$
- $f(x) = \frac{1}{x-5}$
- $f(x) = \frac{1}{\sqrt{x-9}}$
- $f(x) = \frac{1}{\sqrt{x+1}}$

In Exercises 39–46, (a) find the slope of the graph of f at the given point, (b) then find an equation of the tangent line to the graph at the point, and (c) graph the function and the tangent line.

- $f(x) = x^2 - 1$, $(2, 3)$
- $f(x) = 4 - x^2$, $(1, 3)$

41. $f(x) = x^3 - 2x, (1, -1)$ 42. $f(x) = x^3 - x, (2, 6)$
 43. $f(x) = \sqrt{x+1}, (3, 2)$ 44. $f(x) = \sqrt{x-2}, (3, 1)$
 45. $f(x) = \frac{1}{x+5}, (-4, 1)$ 46. $f(x) = \frac{1}{x-3}, (4, 1)$

In Exercises 47–50, use a graphing utility to graph f over the interval $[-2, 2]$ and complete the table. Compare the value of the first derivative with a visual approximation of the slope of the graph.

x	-2	-1.5	-1	-0.5	0	0.5	1	1.5	2
$f(x)$									
$f'(x)$									

47. $f(x) = \frac{1}{2}x^2$ 48. $f(x) = \frac{1}{4}x^3$
 49. $f(x) = \sqrt{x+3}$ 50. $f(x) = \frac{x^2 - 4}{x + 4}$


f In Exercises 51–54, find the derivative of f . Use the derivative to determine any points on the graph of f at which the tangent line is horizontal. Use a graphing utility to verify your results.

51. $f(x) = x^2 - 4x + 3$
 52. $f(x) = x^2 - 6x + 4$
 53. $f(x) = 3x^3 - 9x$
 54. $f(x) = x^3 + 3x$

f In Exercises 55–62, use the function and its derivative to determine any points on the graph of f at which the tangent line is horizontal. Use a graphing utility to verify your results.

55. $f(x) = x^4 - 2x^2, f'(x) = 4x^3 - 4x$
 56. $f(x) = 3x^4 + 4x^3, f'(x) = 12x^3 + 12x^2$
 57. $f(x) = 2 \cos x + x, f'(x) = -2 \sin x + 1$, over the interval $(0, 2\pi)$
 58. $f(x) = x - 2 \sin x, f'(x) = 1 - 2 \cos x$, over the interval $(0, 2\pi)$
 59. $f(x) = x^2 e^x, f'(x) = x^2 e^x + 2x e^x$
 60. $f(x) = x e^{-x}, f'(x) = e^{-x} - x e^{-x}$
 61. $f(x) = x \ln x, f'(x) = \ln x + 1$
 62. $f(x) = \frac{\ln x}{x}, f'(x) = \frac{1 - \ln x}{x^2}$

63. **Population** The projected populations y (in thousands) of New Jersey for selected years from 2010 to 2025 are shown in the table. (Source: U.S. Census Bureau)




Year	Population (in thousands)
2010	9018
2015	9256
2020	9462
2025	9637

Table for 63

- (a) Use the *regression* feature of a graphing utility to find a quadratic model for the data. Let t represent the year, with $t = 10$ corresponding to 2010.
 (b) Use a graphing utility to graph the model found in part (a). Estimate the slope of the graph when $t = 20$, and interpret the result.
 (c) Find the derivative of the model in part (a). Then evaluate the derivative for $t = 20$.
 (d) Write a brief statement regarding your results for parts (a) through (c).

64. **Market Research** The data in the table shows the number N (in thousands) of books sold when the price per book is p (in dollars).



Price, p	Number of books, N
\$10	900
\$15	630
\$20	396
\$25	227
\$30	102
\$35	36

- (a) Use the *regression* feature of a graphing utility to find a quadratic model for the data.
 (b) Use a graphing utility to graph the model found in part (a). Estimate the slopes of the graph when $p = \$15$ and $p = \$30$.
 (c) Use a graphing utility to graph the tangent lines to the model when $p = \$15$ and $p = \$30$. Compare the slopes given by the graphing utility with your estimates in part (b).
 (d) The slopes of the tangent lines at $p = \$15$ and $p = \$30$ are not the same. Explain what this means to the company selling the books.

- f** 65. **Rate of Change** A spherical balloon is inflated. The volume V is approximated by the formula $V(r) = \frac{4}{3}\pi r^3$, where r is the radius.
 (a) Find the derivative of V with respect to r .
 (b) Evaluate the derivative when the radius is 4 inches.

(c) What type of unit would be applied to your answer in part (b)? Explain.

66. Rate of Change An approximately spherical benign tumor is reducing in size. The surface area S is given by the formula $S(r) = 4\pi r^2$, where r is the radius.

- (a) Find the derivative of S with respect to r .
 (b) Evaluate the derivative when the radius is 2 millimeters.
 (c) What type of unit would be applied to your answer in part (b)? Explain.

67. Vertical Motion A water balloon is thrown upward from the top of an 80-foot building with a velocity of 64 feet per second. The height or displacement s (in feet) of the balloon can be modeled by the position function $s(t) = -16t^2 + 64t + 80$, where t is the time in seconds from when it was thrown.

- (a) Find a formula for the instantaneous rate of change of the balloon.
 (b) Find the average rate of change of the balloon after the first three seconds of flight. Explain your results.
 (c) Find the time at which the balloon reaches its maximum height. Explain your method.
 (d) Velocity is given by the derivative of the position function. Find the velocity of the balloon as it impacts the ground.
 (e) Use a graphing utility to graph the model and verify your results for parts (a)–(d).

68. Vertical Motion A Sacajawea dollar is dropped from the top of a 120-foot building. The height or displacement s (in feet) of the coin can be modeled by the position function $s(t) = -16t^2 + 120$, where t is the time in seconds from when it was dropped.

- (a) Find a formula for the instantaneous rate of change of the coin.
 (b) Find the average rate of change of the coin after the first two seconds of free fall. Explain your results.
 (c) Velocity is given by the derivative of the position function. Find the velocity of the coin as it impacts the ground.
 (d) Find the time when the coin's velocity is -60 feet per second.
 (e) Use a graphing utility to graph the model and verify your results for parts (a)–(d).

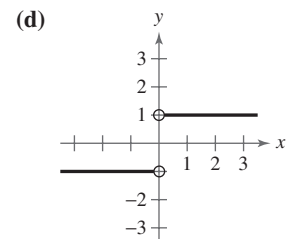
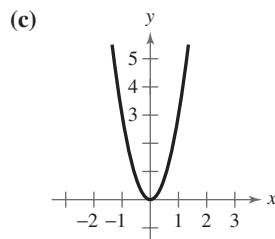
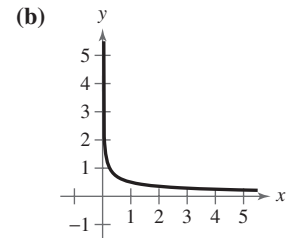
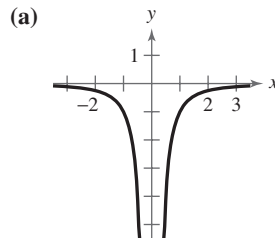
Synthesis

True or False? In Exercises 69 and 70, determine whether the statement is true or false. Justify your answer.

69. The slope of the graph of $y = x^2$ is different at every point on the graph of f .

70. A tangent line to a graph can intersect the graph only at the point of tangency.

Library of Parent Functions In Exercises 71–74, match the function with the graph of its derivative. It is not necessary to find the derivative of the function. [The graphs are labeled (a), (b), (c), and (d).]



71. $f(x) = \sqrt{x}$

72. $f(x) = \frac{1}{x}$

73. $f(x) = |x|$

74. $f(x) = x^3$

75. **Think About It** Sketch the graph of a function whose derivative is always positive.

76. **Think About It** Sketch the graph of a function whose derivative is always negative.

Skills Review

In Exercises 77–80, sketch the graph of the rational function. As sketching aids, check for intercepts, vertical asymptotes, horizontal asymptotes, and slant asymptotes. Use a graphing utility to verify your graph.

77. $f(x) = \frac{1}{x^2 - x - 2}$

78. $f(x) = \frac{x - 2}{x^2 - 4x + 3}$

79. $f(x) = \frac{x^2 - x - 2}{x - 2}$

80. $f(x) = \frac{x^2 - 16}{x + 4}$

In Exercises 81–84, find the cross product of the vectors.

81. $\langle 1, 1, 1 \rangle, \langle 2, 1, -1 \rangle$

82. $\langle -10, 0, 6 \rangle, \langle 7, 0, 0 \rangle$

83. $\langle -4, 10, 0 \rangle, \langle 4, -1, 0 \rangle$

84. $\langle 8, -7, 14 \rangle, \langle -1, 8, 4 \rangle$

11.4 Limits at Infinity and Limits of Sequences

Limits at Infinity and Horizontal Asymptotes

As pointed out at the beginning of this chapter, there are two basic problems in calculus: finding **tangent lines** and finding the **area** of a region. In Section 11.3, you saw how limits can be used to solve the tangent line problem. In this section and the next, you will see how a different type of limit, a *limit at infinity*, can be used to solve the area problem. To get an idea of what is meant by a limit at infinity, consider the function

$$f(x) = \frac{x + 1}{2x}.$$

The graph of f is shown in Figure 11.32. From earlier work, you know that $y = \frac{1}{2}$ is a horizontal asymptote of the graph of this function. Using limit notation, this can be written as follows.

$$\lim_{x \rightarrow -\infty} f(x) = \frac{1}{2} \quad \text{Horizontal asymptote to the left}$$

$$\lim_{x \rightarrow \infty} f(x) = \frac{1}{2} \quad \text{Horizontal asymptote to the right}$$

These limits mean that the value of $f(x)$ gets arbitrarily close to $\frac{1}{2}$ as x decreases or increases without bound.

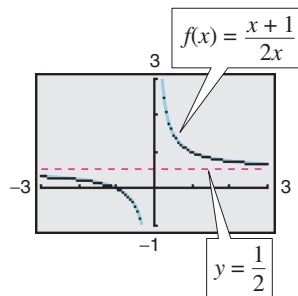


Figure 11.32

Definition of Limits at Infinity

If f is a function and L_1 and L_2 are real numbers, the statements

$$\lim_{x \rightarrow -\infty} f(x) = L_1 \quad \text{Limit as } x \text{ approaches } -\infty$$

and

$$\lim_{x \rightarrow \infty} f(x) = L_2 \quad \text{Limit as } x \text{ approaches } \infty$$

denote the **limits at infinity**. The first statement is read “the limit of $f(x)$ as x approaches $-\infty$ is L_1 ,” and the second is read “the limit of $f(x)$ as x approaches ∞ is L_2 .”

What you should learn

- Evaluate limits of functions at infinity.
- Find limits of sequences.

Why you should learn it

Finding limits at infinity is useful in highway safety applications. For instance, in Exercise 56 on page 819, you are asked to find a limit at infinity to predict the number of injuries due to motor vehicle accidents in the United States.



AP Photos

TECHNOLOGY TIP

Recall from Section 2.7 that some graphing utilities have difficulty graphing rational functions. In this text, rational functions are graphed using the *dot* mode of a graphing utility, and a blue curve is placed behind the graphing utility’s display to indicate where the graph should appear.

To help evaluate limits at infinity, you can use the following definition.

Limits at Infinity

If r is a positive real number, then

$$\lim_{x \rightarrow \infty} \frac{1}{x^r} = 0. \quad \text{Limit toward the right}$$

Furthermore, if x^r is defined when $x < 0$, then

$$\lim_{x \rightarrow -\infty} \frac{1}{x^r} = 0. \quad \text{Limit toward the left}$$

Limits at infinity share many of the properties of limits listed in Section 11.1. Some of these properties are demonstrated in the next example.

Example 1 Evaluating a Limit at Infinity

Find the limit.

$$\lim_{x \rightarrow \infty} \left(4 - \frac{3}{x^2} \right)$$

Algebraic Solution

Use the properties of limits listed in Section 11.1.

$$\begin{aligned} \lim_{x \rightarrow \infty} \left(4 - \frac{3}{x^2} \right) &= \lim_{x \rightarrow \infty} 4 - \lim_{x \rightarrow \infty} \frac{3}{x^2} \\ &= \lim_{x \rightarrow \infty} 4 - 3 \left(\lim_{x \rightarrow \infty} \frac{1}{x^2} \right) \\ &= 4 - 3(0) \\ &= 4 \end{aligned}$$

So, the limit of $f(x) = 4 - \frac{3}{x^2}$ as x approaches ∞ is 4.

 **CHECKPOINT** Now try Exercise 9.

In Figure 11.33, it appears that the line $y = 4$ is also a horizontal asymptote to the left. You can verify this by showing that

$$\lim_{x \rightarrow -\infty} \left(4 - \frac{3}{x^2} \right) = 4.$$

The graph of a rational function need not have a horizontal asymptote. If it does, however, its left and right asymptotes must be the same.

When evaluating limits at infinity for more complicated rational functions, divide the numerator and denominator by the *highest-powered term* in the denominator. This enables you to evaluate each limit using the limits at infinity at the top of this page.

Exploration

Use a graphing utility to graph the two functions given by

$$y_1 = \frac{1}{\sqrt{x}} \quad \text{and} \quad y_2 = \frac{1}{\sqrt[3]{x}}$$

in the same viewing window. Why doesn't y_1 appear to the left of the y -axis? How does this relate to the statement at the left about the infinite limit

$$\lim_{x \rightarrow -\infty} \frac{1}{x^r} ?$$

Graphical Solution

Use a graphing utility to graph $y = 4 - 3/x^2$. Then use the *trace* feature to determine that as x gets larger and larger, y gets closer and closer to 4, as shown in Figure 11.33. Note that the line $y = 4$ is a horizontal asymptote to the right.

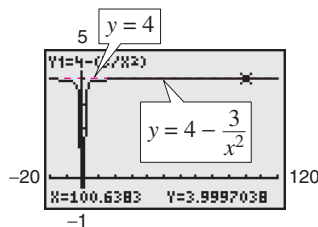


Figure 11.33

Example 2 Comparing Limits at Infinity

Find the limit as x approaches ∞ for each function.

$$\text{a. } f(x) = \frac{-2x + 3}{3x^2 + 1} \quad \text{b. } f(x) = \frac{-2x^2 + 3}{3x^2 + 1} \quad \text{c. } f(x) = \frac{-2x^3 + 3}{3x^2 + 1}$$

Solution

In each case, begin by dividing both the numerator and denominator by x^2 , the highest-powered term in the denominator.

$$\begin{aligned} \text{a. } \lim_{x \rightarrow \infty} \frac{-2x + 3}{3x^2 + 1} &= \lim_{x \rightarrow \infty} \frac{-\frac{2}{x} + \frac{3}{x^2}}{3 + \frac{1}{x^2}} \\ &= \frac{-0 + 0}{3 + 0} \\ &= 0 \end{aligned}$$

$$\begin{aligned} \text{b. } \lim_{x \rightarrow \infty} \frac{-2x^2 + 3}{3x^2 + 1} &= \lim_{x \rightarrow \infty} \frac{-2 + \frac{3}{x^2}}{3 + \frac{1}{x^2}} \\ &= \frac{-2 + 0}{3 + 0} \\ &= -\frac{2}{3} \end{aligned}$$

$$\text{c. } \lim_{x \rightarrow \infty} \frac{-2x^3 + 3}{3x^2 + 1} = \lim_{x \rightarrow \infty} \frac{-2x + \frac{3}{x^2}}{3 + \frac{1}{x^2}}$$

As $3/x^2$ in the numerator approaches 0, the numerator approaches $-\infty$. In this case, you can conclude that the limit does not exist because the numerator decreases without bound as the denominator approaches 3.

 **CHECKPOINT** Now try Exercise 15.

In Example 2, observe that when the degree of the numerator is less than the degree of the denominator, as in part (a), the limit is 0. When the degrees of the numerator and denominator are equal, as in part (b), the limit is the ratio of the coefficients of the highest-powered terms. When the degree of the numerator is greater than the degree of the denominator, as in part (c), the limit does not exist.

This result seems reasonable when you realize that for large values of x , the highest-powered term of a polynomial is the most “influential” term. That is, a polynomial tends to behave as its highest-powered term behaves as x approaches positive or negative infinity.

Exploration

Use a graphing utility to complete the table below to verify that

$$\lim_{x \rightarrow \infty} \frac{1}{x} = 0.$$

x	10^0	10^1	10^2
$\frac{1}{x}$			

x	10^3	10^4	10^5
$\frac{1}{x}$			

Make a conjecture about

$$\lim_{x \rightarrow 0} \frac{1}{x}.$$

Limits at Infinity for Rational Functions

Consider the rational function $f(x) = N(x)/D(x)$, where

$$N(x) = a_n x^n + \cdots + a_0 \quad \text{and} \quad D(x) = b_m x^m + \cdots + b_0.$$

The limit of $f(x)$ as x approaches positive or negative infinity is as follows.

$$\lim_{x \rightarrow \pm\infty} f(x) = \begin{cases} 0, & n < m \\ \frac{a_n}{b_m}, & n = m \end{cases}$$

If $n > m$, the limit does not exist.

Example 3 Finding the Average Cost



You are manufacturing greeting cards that cost \$0.50 per card to produce. Your initial investment is \$5000, which implies that the total cost C of producing x cards is given by $C = 0.50x + 5000$. The average cost \bar{C} per card is given by

$$\bar{C} = \frac{C}{x} = \frac{0.50x + 5000}{x}.$$

Find the average cost per card when (a) $x = 1000$, (b) $x = 10,000$, and (c) $x = 100,000$. (d) What is the limit of \bar{C} as x approaches infinity?

Solution

a. When $x = 1000$, the average cost per card is

$$\begin{aligned} \bar{C} &= \frac{0.50(1000) + 5000}{1000} && x = 1000 \\ &= \$5.50. \end{aligned}$$

b. When $x = 10,000$, the average cost per card is

$$\begin{aligned} \bar{C} &= \frac{0.50(10,000) + 5000}{10,000} && x = 10,000 \\ &= \$1.00. \end{aligned}$$

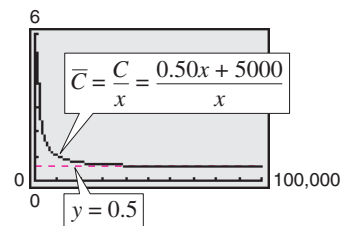
c. When $x = 100,000$, the average cost per card is

$$\begin{aligned} \bar{C} &= \frac{0.50(100,000) + 5000}{100,000} && x = 100,000 \\ &= \$0.55. \end{aligned}$$

d. As x approaches infinity, the limit of \bar{C} is

$$\lim_{x \rightarrow \infty} \frac{0.50x + 5000}{x} = \$0.50. \quad x \rightarrow \infty$$

The graph of \bar{C} is shown in Figure 11.34.



As $x \rightarrow \infty$, the average cost per card approaches \$0.50.

Figure 11.34

CHECKPOINT Now try Exercise 53.

Limits of Sequences

Limits of sequences have many of the same properties as limits of functions. For instance, consider the sequence whose n th term is $a_n = 1/2^n$.

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \dots$$

As n increases without bound, the terms of this sequence get closer and closer to 0, and the sequence is said to **converge** to 0. Using limit notation, you can write

$$\lim_{n \rightarrow \infty} \frac{1}{2^n} = 0.$$

The following relationship shows how limits of functions of x can be used to evaluate the limit of a sequence.

Limit of a Sequence

Let f be a function of a real variable such that

$$\lim_{x \rightarrow \infty} f(x) = L.$$

If $\{a_n\}$ is a sequence such that $f(n) = a_n$ for every positive integer n , then

$$\lim_{n \rightarrow \infty} a_n = L.$$

A sequence that does not converge is said to **diverge**. For instance, the sequence $1, -1, 1, -1, 1, \dots$ diverges because it does not approach a unique number.

Example 4 Finding the Limit of a Sequence

Find the limit of each sequence. (Assume n begins with 1.)

a. $a_n = \frac{2n + 1}{n + 4}$ b. $b_n = \frac{2n + 1}{n^2 + 4}$ c. $c_n = \frac{2n^2 + 1}{4n^2}$

Solution

a. $\lim_{n \rightarrow \infty} \frac{2n + 1}{n + 4} = 2$ $\frac{3}{5}, \frac{5}{6}, \frac{7}{7}, \frac{9}{8}, \frac{11}{9}, \frac{13}{10}, \dots \rightarrow 2$

b. $\lim_{n \rightarrow \infty} \frac{2n + 1}{n^2 + 4} = 0$ $\frac{3}{5}, \frac{5}{8}, \frac{7}{13}, \frac{9}{20}, \frac{11}{29}, \frac{13}{40}, \dots \rightarrow 0$

c. $\lim_{n \rightarrow \infty} \frac{2n^2 + 1}{4n^2} = \frac{1}{2}$ $\frac{3}{4}, \frac{9}{16}, \frac{19}{36}, \frac{33}{64}, \frac{51}{100}, \frac{73}{144}, \dots \rightarrow \frac{1}{2}$

 **CHECKPOINT** Now try Exercise 39.

TECHNOLOGY TIP

There are a number of ways to use a graphing utility to generate the terms of a sequence. For instance, you can display the first 10 terms of the sequence

$$a_n = \frac{1}{2^n}$$

using the *sequence* feature or the *table* feature. For instructions on how to use the *sequence* feature and the *table* feature, see Appendix A; for specific keystrokes, go to this textbook's *Online Study Center*.

STUDY TIP

You can use the definition of limits at infinity for rational functions on page 814 to verify the limits of the sequences in Example 4.

In the next section, you will encounter limits of sequences such as that shown in Example 5. A strategy for evaluating such limits is to begin by writing the n th term in standard rational function form. Then you can determine the limit by comparing the degrees of the numerator and denominator, as shown on page 814.

Example 5 Finding the Limit of a Sequence

Find the limit of the sequence whose n th term is

$$a_n = \frac{8}{n^3} \left[\frac{n(n+1)(2n+1)}{6} \right].$$

Algebraic Solution

Begin by writing the n th term in standard rational function form—as the ratio of two polynomials.

$$a_n = \frac{8}{n^3} \left[\frac{n(n+1)(2n+1)}{6} \right] \quad \text{Write original } n\text{th term.}$$

$$= \frac{8(n)(n+1)(2n+1)}{6n^3} \quad \text{Multiply fractions.}$$

$$= \frac{8n^3 + 12n^2 + 4n}{3n^3} \quad \text{Write in standard rational form.}$$

From this form, you can see that the degree of the numerator is equal to the degree of the denominator. So, the limit of the sequence is the ratio of the coefficients of the highest-powered terms.

$$\lim_{n \rightarrow \infty} \frac{8n^3 + 12n^2 + 4n}{3n^3} = \frac{8}{3}$$

 **CHECKPOINT** Now try Exercise 49.

Numerical Solution

Enter the sequence into a graphing utility. Be sure the graphing utility is set to *sequence* mode. Then use the *table* feature (in *ask* mode) to create a table that shows the values of a_n as n becomes larger and larger, as shown in Figure 11.35.

n	$a(n)$
1	8
10	3.08
100	2.7068
1000	2.6707
10000	2.6671

Figure 11.35

From the table, you can estimate that as n approaches ∞ , a_n gets closer and closer to $2.667 \approx \frac{8}{3}$.

Exploration

In the table in Example 5 above, the value of a_n approaches its limit of $\frac{8}{3}$ rather slowly. (The first term to be accurate to three decimal places is $a_{4801} \approx 2.667$.) Each of the following sequences converges to 0. Which converges the quickest? Which converges the slowest? Why? Write a short paragraph discussing your conclusions.

a. $a_n = \frac{1}{n}$ b. $b_n = \frac{1}{n^2}$ c. $c_n = \frac{1}{2^n}$

d. $d_n = \frac{1}{n!}$ e. $h_n = \frac{2^n}{n!}$

11.4 Exercises

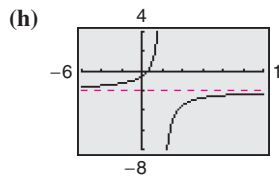
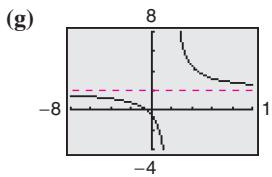
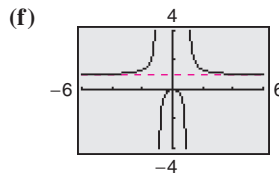
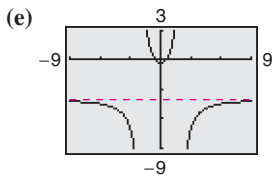
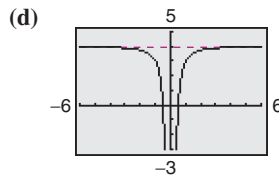
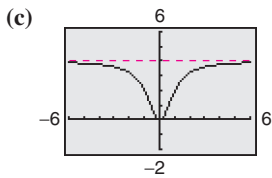
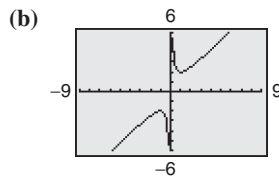
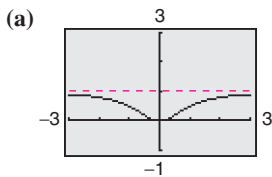
See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Vocabulary Check

Fill in the blanks.

1. A _____ at _____ can be used to solve the area problem in calculus.
2. A sequence that has a limit is said to _____.
3. A sequence that does not have a limit is said to _____.

In Exercises 1–8, match the function with its graph, using horizontal asymptotes as aids. [The graphs are labeled (a), (b), (c), (d), (e), (f), (g), and (h).]



1. $f(x) = \frac{4x^2}{x^2 + 1}$

2. $f(x) = \frac{x^2}{x^2 + 1}$

3. $f(x) = 4 - \frac{1}{x^2}$

4. $f(x) = x + \frac{1}{x}$

5. $f(x) = \frac{x^2}{x^2 - 1}$

6. $f(x) = \frac{2x + 1}{x - 2}$

7. $f(x) = \frac{1 - 2x}{x - 2}$

8. $f(x) = \frac{1 - 4x^2}{x^2 - 4}$

In Exercises 9–28, find the limit (if it exists). If the limit does not exist, explain why. Use a graphing utility to verify your result graphically.

9. $\lim_{x \rightarrow \infty} \frac{3}{x^2}$

10. $\lim_{x \rightarrow \infty} \frac{5}{2x}$

11. $\lim_{x \rightarrow \infty} \frac{3 + x}{3 - x}$

12. $\lim_{x \rightarrow \infty} \frac{2 - 7x}{2 + 3x}$

13. $\lim_{x \rightarrow -\infty} \frac{5x - 2}{6x + 1}$

14. $\lim_{x \rightarrow -\infty} \frac{5 - 3x}{x + 4}$

15. $\lim_{x \rightarrow -\infty} \frac{4x^2 - 3}{2 - x^2}$

16. $\lim_{x \rightarrow -\infty} \frac{x^2 + 3}{5x^2 - 4}$

17. $\lim_{t \rightarrow \infty} \frac{t^2}{t + 3}$

18. $\lim_{y \rightarrow \infty} \frac{4y^4}{y^2 + 3}$

19. $\lim_{t \rightarrow \infty} \frac{4t^2 + 3t - 1}{3t^2 + 2t - 5}$

20. $\lim_{x \rightarrow \infty} \frac{5 - 6x - 3x^2}{2x^2 + x + 4}$

21. $\lim_{y \rightarrow -\infty} \frac{3 + 8y - 4y^2}{3 - y - 2y^2}$

22. $\lim_{t \rightarrow -\infty} \frac{t^2 + 9t - 10}{2 + 4t - 3t^2}$

23. $\lim_{x \rightarrow -\infty} \frac{-(x^2 + 3)}{(2 - x)^2}$

24. $\lim_{x \rightarrow \infty} \frac{2x^2 - 6}{(x - 1)^2}$

25. $\lim_{x \rightarrow -\infty} \left[\frac{x}{(x + 1)^2} - 4 \right]$

26. $\lim_{x \rightarrow \infty} \left[7 + \frac{2x^2}{(x + 3)^2} \right]$

27. $\lim_{t \rightarrow \infty} \left(\frac{1}{3t^2} - \frac{5t}{t + 2} \right)$

28. $\lim_{x \rightarrow \infty} \left[\frac{x}{2x + 1} + \frac{3x^2}{(x - 3)^2} \right]$

In Exercises 29–34, use a graphing utility to graph the function and verify that the horizontal asymptote corresponds to the limit at infinity.

29. $y = \frac{3x}{1 - x}$

30. $y = \frac{x^2}{x^2 + 4}$

31. $y = \frac{2x}{1 - x^2}$

32. $y = \frac{2x + 1}{x^2 - 1}$

33. $y = 1 - \frac{3}{x^2}$

34. $y = 2 + \frac{1}{x}$

Numerical and Graphical Analysis In Exercises 35–38, (a) complete the table and numerically estimate the limit as x approaches infinity and (b) use a graphing utility to graph the function and estimate the limit graphically.

n	10^0	10^1	10^2	10^3	10^4	10^5	10^6
$f(x)$							

35. $f(x) = x - \sqrt{x^2 + 2}$ 36. $f(x) = 3x - \sqrt{9x^2 + 1}$
 37. $f(x) = 3(2x - \sqrt{4x^2 + x})$
 38. $f(x) = 4(4x - \sqrt{16x^2 - x})$

In Exercises 39–48, write the first five terms of the sequence and find the limit of the sequence (if it exists). If the limit does not exist, explain why. Assume n begins with 1.

39. $a_n = \frac{n + 1}{n^2 + 1}$ 40. $a_n = \frac{n}{n^2 + 1}$
 41. $a_n = \frac{n}{2n + 1}$ 42. $a_n = \frac{4n - 1}{n + 3}$
 43. $a_n = \frac{n^2}{3n + 2}$ 44. $a_n = \frac{4n^2 + 1}{2n}$
 45. $a_n = \frac{(n + 1)!}{n!}$ 46. $a_n = \frac{(3n - 1)!}{(3n + 1)!}$
 47. $a_n = \frac{(-1)^n}{n}$ 48. $a_n = \frac{(-1)^{n+1}}{n^2}$

In Exercises 49–52, use a graphing utility to complete the table and estimate the limit of the sequence as n approaches infinity. Then find the limit algebraically.

n	10^0	10^1	10^2	10^3	10^4	10^5	10^6
a_n							

49. $a_n = \frac{1}{n} \left(n + \frac{1}{n} \left[\frac{n(n + 1)}{2} \right] \right)$
 50. $a_n = \frac{4}{n} \left(n + \frac{4}{n} \left[\frac{n(n + 1)}{2} \right] \right)$
 51. $a_n = \frac{16}{n^3} \left[\frac{n(n + 1)(2n + 1)}{6} \right]$
 52. $a_n = \frac{n(n + 1)}{n^2} - \frac{1}{n^4} \left[\frac{n(n + 1)}{2} \right]^2$


53. **Average Cost** The cost function for a certain model of a personal digital assistant (PDA) is given by $C = 13.50x + 45,750$, where C is the cost (in dollars) and x is the number of PDAs produced.
 (a) Write a model for the average cost per unit produced.
 (b) Find the average costs per unit when $x = 100$ and $x = 1000$.

- (c) Determine the limit of the average cost function as x approaches infinity. Explain the meaning of the limit in the context of the problem.

54. **Average Cost** The cost function for a company to recycle x tons of material is given by $C = 1.25x + 10,500$, where C is the cost (in dollars).

- (a) Write a model for the average cost per ton of material recycled.
 (b) Find the average costs of recycling 100 tons of material and 1000 tons of material.
 (c) Determine the limit of the average cost function as x approaches infinity. Explain the meaning of the limit in the context of the problem.

55. **School Enrollment** The table shows the school enrollments E (in millions) in the United States for the years 1990 through 2003. (Source: U.S. National Center for Education Statistics)



Year	Enrollment, E (in millions)
1990	60.3
1991	61.7
1992	62.6
1993	63.1
1994	63.9
1995	64.8
1996	65.7
1997	66.5
1998	67.0
1999	67.7
2000	68.7
2001	69.9
2002	71.2
2003	71.4


A model for the data is given by

$$E(t) = \frac{0.702t^2 + 61.49}{0.009t^2 + 1}, \quad 0 \leq t \leq 13$$

where t represents the year, with $t = 0$ corresponding to 1990.

- (a) Use a graphing utility to create a scatter plot of the data and graph the model in the same viewing window. How do they compare?
 (b) Use the model to predict the enrollments in 2004 and 2008.
 (c) Find the limit of the model as $t \rightarrow \infty$ and interpret its meaning in the context of the situation.
 (d) Can the model be used to predict the enrollments for future years? Explain.

- 56. Highway Safety** The table shows the numbers of injuries N (in thousands) from motor vehicle accidents in the United States for the years 1991 through 2004. (Source: U.S. National Highway Safety Administration)



Year	Injuries, N (in thousands)
1991	3097
1992	3070
1993	3149
1994	3266
1995	3465
1996	3483
1997	3348
1998	3192
1999	3236
2000	3189
2001	3033
2002	2926
2003	2889
2004	2788

A model for the data is given by

$$N(t) = \frac{40.8189t^2 - 500.059t + 2950.8}{0.0157t^2 - 0.192t + 1}, \quad 1 \leq t \leq 14$$

where t represents the year, with $t = 1$ corresponding to 1991.

- Use a graphing utility to create a scatter plot of the data and graph the model in the same viewing window. How do they compare?
- Use the model to predict the numbers of injuries in 2006 and 2010.
- Find the limit of the model as $t \rightarrow \infty$ and interpret its meaning in the context of the situation.
- Can the model be used to predict the numbers of injuries for future years? Explain.

Synthesis

True or False? In Exercises 57–60, determine whether the statement is true or false. Justify your answer.

- Every rational function has a horizontal asymptote.
- If $f(x)$ increases without bound as x approaches c , then the limit of $f(x)$ exists.
- If a sequence converges, then it has a limit.
- When the degrees of the numerator and denominator of a rational function are equal, the limit as x goes to infinity does not exist.

- Think About It** Find the functions f and g such that both $f(x)$ and $g(x)$ increase without bound as x approaches c , but $\lim_{x \rightarrow c} [f(x) - g(x)] \neq \infty$.
- Think About It** Use a graphing utility to graph the function $f(x) = x/\sqrt{x^2 + 1}$. How many horizontal asymptotes does the function appear to have? What are the horizontal asymptotes?

Exploration In Exercises 63–66, use a graphing utility to create a scatter plot of the terms of the sequence. Determine whether the sequence converges or diverges. If it converges, estimate its limit.

63. $a_n = 4\left(\frac{2}{3}\right)^n$

64. $a_n = 3\left(\frac{3}{2}\right)^n$

65. $a_n = \frac{3[1 - (1.5)^n]}{1 - 1.5}$

66. $a_n = \frac{3[1 - (0.5)^n]}{1 - 0.5}$

Skills Review

In Exercises 67 and 68, sketch the graphs of y and each transformation on the same rectangular coordinate system.

67. $y = x^4$

(a) $f(x) = (x + 3)^4$

(b) $f(x) = x^4 - 1$

(c) $f(x) = -2 + x^4$

(d) $f(x) = \frac{1}{2}(x - 4)^4$

68. $y = x^3$

(a) $f(x) = (x + 2)^3$

(b) $f(x) = 3 + x^3$

(c) $f(x) = 2 - \frac{1}{4}x^3$

(d) $f(x) = 3(x + 1)^3$

In Exercises 69–72, divide using long division.

69. $(x^4 + 2x^3 - 3x^2 - 8x - 4) \div (x^2 - 4)$

70. $(2x^5 - 8x^3 + 4x - 1) \div (x^2 - 2x + 1)$

71. $(3x^4 + 17x^3 + 10x^2 - 9x - 8) \div (3x + 2)$

72. $(10x^3 + 51x^2 + 48x - 28) \div (5x - 2)$

In Exercises 73–76, find all the real zeros of the polynomial function. Use a graphing utility to graph the function and verify that the real zeros are the x -intercepts of the graph of the function.

73. $f(x) = x^4 - x^3 - 20x^2$

74. $f(x) = x^5 + x^3 - 6x$

75. $f(x) = x^3 - 3x^2 + 2x - 6$

76. $f(x) = x^3 - 4x^2 - 25x + 100$

In Exercises 77–80, find the sum.

77. $\sum_{i=1}^6 (2i + 3)$

78. $\sum_{i=0}^4 5i^2$

79. $\sum_{k=1}^{10} 15$

80. $\sum_{k=0}^8 \frac{3}{k^2 + 1}$

11.5 The Area Problem

Limits of Summations

Earlier in the text, you used the concept of a limit to obtain a formula for the sum S of an infinite geometric series

$$S = a_1 + a_1r + a_1r^2 + \cdots = \sum_{i=1}^{\infty} a_1r^{i-1} = \frac{a_1}{1-r}, \quad |r| < 1.$$

Using limit notation, this sum can be written as

$$S = \lim_{n \rightarrow \infty} \sum_{i=1}^n a_1r^{i-1} = \lim_{n \rightarrow \infty} \frac{a_1(1-r^n)}{1-r} = \frac{a_1}{1-r}.$$

The following summation formulas and properties are used to evaluate finite and infinite summations.

Summation Formulas and Properties

- $\sum_{i=1}^n c = cn$, c is a constant.
- $\sum_{i=1}^n i = \frac{n(n+1)}{2}$
- $\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$
- $\sum_{i=1}^n i^3 = \frac{n^2(n+1)^2}{4}$
- $\sum_{i=1}^n (a_i \pm b_i) = \sum_{i=1}^n a_i \pm \sum_{i=1}^n b_i$
- $\sum_{i=1}^n ka_i = k \sum_{i=1}^n a_i$, k is a constant.

What you should learn

- Find limits of summations.
- Use rectangles to approximate areas of plane regions.
- Use limits of summations to find areas of plane regions.

Why you should learn it

Limits of summations are useful in determining areas of plane regions. For instance, in Exercise 46 on page 827, you are asked to find the limit of a summation to determine the area of a parcel of land bounded by a stream and two roads.



Adam Woolfitt/Corbis

Example 1 Evaluating a Summation

Evaluate the summation.

$$\sum_{i=1}^{200} i = 1 + 2 + 3 + 4 + \cdots + 200$$

Solution

Using Formula 2 with $n = 200$, you can write

$$\begin{aligned} \sum_{i=1}^n i &= \frac{n(n+1)}{2} \\ \sum_{i=1}^{200} i &= \frac{200(200+1)}{2} \\ &= \frac{40,200}{2} \\ &= 20,100. \end{aligned}$$

Prerequisite Skills

Recall that the sum of a finite geometric sequence is given by

$$\sum_{i=1}^n a_1r^{i-1} = a_1 \left(\frac{1-r^n}{1-r} \right).$$

Furthermore, if $0 < |r| < 1$, then $r^n \rightarrow 0$ as $n \rightarrow \infty$. To review sums of geometric sequences, see Section 8.3.

 **CHECKPOINT** Now try Exercise 1.

Example 2 Evaluating a Summation

Evaluate the summation

$$S = \sum_{i=1}^n \frac{i+2}{n^2} = \frac{3}{n^2} + \frac{4}{n^2} + \frac{5}{n^2} + \cdots + \frac{n+2}{n^2}$$

for $n = 10, 100, 1000,$ and $10,000.$

Solution

Begin by applying summation formulas and properties to simplify $S.$ In the second line of the solution, note that $1/n^2$ can be factored out of the sum because n is considered to be constant. You could not factor i out of the summation because i is the (variable) index of summation.

$$\begin{aligned} S &= \sum_{i=1}^n \frac{i+2}{n^2} && \text{Write original form of summation.} \\ &= \frac{1}{n^2} \sum_{i=1}^n (i+2) && \text{Factor constant } 1/n^2 \text{ out of sum.} \\ &= \frac{1}{n^2} \left(\sum_{i=1}^n i + \sum_{i=1}^n 2 \right) && \text{Write as two sums.} \\ &= \frac{1}{n^2} \left[\frac{n(n+1)}{2} + 2n \right] && \text{Apply Formulas 1 and 2.} \\ &= \frac{1}{n^2} \left(\frac{n^2 + 5n}{2} \right) && \text{Add fractions.} \\ &= \frac{n+5}{2n} && \text{Simplify.} \end{aligned}$$

Now you can evaluate the sum by substituting the appropriate values of $n,$ as shown in the following table.

n	10	100	1000	10,000
$\sum_{i=1}^n \frac{i+2}{n^2} = \frac{n+5}{2n}$	0.75	0.525	0.5025	0.50025



CHECKPOINT Now try Exercise 11.

In Example 2, note that the sum appears to approach a limit as n increases. To find the limit of $(n+5)/2n$ as n approaches infinity, you can use the techniques from Section 11.4 to write

$$\lim_{n \rightarrow \infty} \frac{n+5}{2n} = \frac{1}{2}.$$

TECHNOLOGY TIP

Some graphing utilities have a *sum sequence* feature that is useful for computing summations. For instructions on how to use the *sum sequence* feature, see Appendix A; for specific keystrokes, go to this textbook's *Online Study Center.*

Be sure you notice the strategy used in Example 2. Rather than separately evaluating the sums

$$\sum_{i=1}^{10} \frac{i+2}{n^2}, \quad \sum_{i=1}^{100} \frac{i+2}{n^2}, \quad \sum_{i=1}^{1000} \frac{i+2}{n^2}, \quad \sum_{i=1}^{10,000} \frac{i+2}{n^2}$$

it was more efficient first to convert to rational form using the summation formulas and properties listed on page 820.

$$S = \underbrace{\sum_{i=1}^n \frac{i+2}{n^2}}_{\text{Summation form}} = \underbrace{\frac{n+5}{2n}}_{\text{Rational form}}$$

With this rational form, each sum can be evaluated by simply substituting appropriate values of n .

Example 3 Finding the Limit of a Summation

Find the limit of $S(n)$ as $n \rightarrow \infty$.

$$S(n) = \sum_{i=1}^n \left(1 + \frac{i}{n}\right)^2 \left(\frac{1}{n}\right)$$

Solution

Begin by rewriting the summation in rational form.

$$\begin{aligned} S(n) &= \sum_{i=1}^n \left(1 + \frac{i}{n}\right)^2 \left(\frac{1}{n}\right) \\ &= \sum_{i=1}^n \left(\frac{n^2 + 2ni + i^2}{n^2}\right) \left(\frac{1}{n}\right) \\ &= \frac{1}{n^3} \sum_{i=1}^n (n^2 + 2ni + i^2) \\ &= \frac{1}{n^3} \left(\sum_{i=1}^n n^2 + \sum_{i=1}^n 2ni + \sum_{i=1}^n i^2 \right) \\ &= \frac{1}{n^3} \left\{ n^3 + 2n \left[\frac{n(n+1)}{2} \right] + \frac{n(n+1)(2n+1)}{6} \right\} \\ &= \frac{14n^3 + 9n^2 + n}{6n^3} \end{aligned}$$

In this rational form, you can now find the limit as $n \rightarrow \infty$.

$$\begin{aligned} \lim_{n \rightarrow \infty} S(n) &= \lim_{n \rightarrow \infty} \frac{14n^3 + 9n^2 + n}{6n^3} \\ &= \frac{14}{6} = \frac{7}{3} \end{aligned}$$

STUDY TIP

As you can see from Example 3, there is a lot of algebra involved in rewriting a summation in rational form. You may want to review simplifying rational expressions if you are having difficulty with this procedure.

Write original form of summation.

Square $(1 + i/n)$ and write as a single fraction.

Factor constant $1/n^3$ out of the sum.

Write as three sums.

Use summation formulas.

Simplify.

Prerequisite Skills

To review limits at infinity, see Section 11.4.



Now try Exercise 13.

The Area Problem

You now have the tools needed to solve the second basic problem of calculus: the area problem. The problem is to find the *area* of the region R bounded by the graph of a nonnegative, continuous function f , the x -axis, and the vertical lines $x = a$ and $x = b$, as shown in Figure 11.36.

If the region R is a square, a triangle, a trapezoid, or a semicircle, you can find its area by using a geometric formula. For more general regions, however, you must use a different approach—one that involves the limit of a summation. The basic strategy is to use a collection of rectangles of equal width that approximates the region R , as illustrated in Example 4.

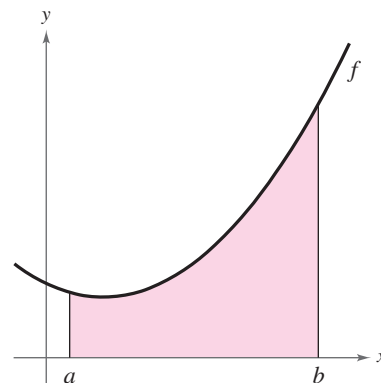


Figure 11.36

Example 4 Approximating the Area of a Region

Use the five rectangles in Figure 11.37 to approximate the area of the region bounded by the graph of $f(x) = 6 - x^2$, the x -axis, and the lines $x = 0$ and $x = 2$.

Solution

Because the length of the interval along the x -axis is 2 and there are five rectangles, the width of each rectangle is $\frac{2}{5}$. The height of each rectangle can be obtained by evaluating f at the right endpoint of each interval. The five intervals are as follows.

$$\left[0, \frac{2}{5}\right], \quad \left[\frac{2}{5}, \frac{4}{5}\right], \quad \left[\frac{4}{5}, \frac{6}{5}\right], \quad \left[\frac{6}{5}, \frac{8}{5}\right], \quad \left[\frac{8}{5}, \frac{10}{5}\right]$$

Notice that the right endpoint of each interval is $\frac{2i}{5}$ for $i = 1, 2, 3, 4,$ and 5 . The sum of the areas of the five rectangles is

$$\begin{aligned} \sum_{i=1}^5 \overbrace{f\left(\frac{2i}{5}\right)}^{\text{Height}} \overbrace{\left(\frac{2}{5}\right)}^{\text{Width}} &= \sum_{i=1}^5 \left[6 - \left(\frac{2i}{5}\right)^2\right] \left(\frac{2}{5}\right) \\ &= \frac{2}{5} \left(\sum_{i=1}^5 6 - \frac{4}{25} \sum_{i=1}^5 i^2 \right) \\ &= \frac{2}{5} \left(30 - \frac{44}{5} \right) = \frac{212}{25} = 8.48. \end{aligned}$$

So, you can approximate the area of R as 8.48 square units.

CHECKPOINT Now try Exercise 19.

By increasing the number of rectangles used in Example 4, you can obtain closer and closer approximations of the area of the region. For instance, using 25 rectangles of width $\frac{2}{25}$ each, you can approximate the area to be $A \approx 9.17$ square units. The following table shows even better approximations.

n	5	25	100	1000	5000
Approximate area	8.48	9.17	9.29	9.33	9.33

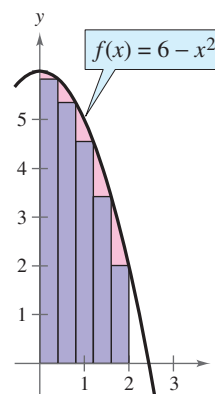


Figure 11.37

Based on the procedure illustrated in Example 4, the *exact area of a plane region* R is given by the limit of the sum of n rectangles as n approaches ∞ .

Area of a Plane Region

Let f be continuous and nonnegative on the interval $[a, b]$. The **area** A of the region bounded by the graph of f , the x -axis, and the vertical lines $x = a$ and $x = b$ is given by

$$A = \lim_{n \rightarrow \infty} \sum_{i=1}^n \underbrace{f\left(a + \frac{(b-a)i}{n}\right)}_{\text{Height}} \underbrace{\left(\frac{b-a}{n}\right)}_{\text{Width}}$$

Example 5 Finding the Area of a Region

Find the area of the region bounded by the graph of $f(x) = x^2$ and the x -axis between $x = 0$ and $x = 1$, as shown in Figure 11.38.

Solution

Begin by finding the dimensions of the rectangles.

$$\text{Width: } \frac{b-a}{n} = \frac{1-0}{n} = \frac{1}{n}$$

$$\text{Height: } f\left(a + \frac{(b-a)i}{n}\right) = f\left(0 + \frac{(1-0)i}{n}\right) = f\left(\frac{i}{n}\right) = \frac{i^2}{n^2}$$

Next, approximate the area as the sum of the areas of n rectangles.

$$\begin{aligned} A &\approx \sum_{i=1}^n f\left(a + \frac{(b-a)i}{n}\right) \left(\frac{b-a}{n}\right) \\ &= \sum_{i=1}^n \left(\frac{i^2}{n^2}\right) \left(\frac{1}{n}\right) \\ &= \sum_{i=1}^n \frac{i^2}{n^3} \\ &= \frac{1}{n^3} \sum_{i=1}^n i^2 \\ &= \frac{1}{n^3} \left[\frac{n(n+1)(2n+1)}{6} \right] \\ &= \frac{2n^3 + 3n^2 + n}{6n^3} \end{aligned}$$

Finally, find the exact area by taking the limit as n approaches ∞ .

$$A = \lim_{n \rightarrow \infty} \frac{2n^3 + 3n^2 + n}{6n^3} = \frac{1}{3}$$

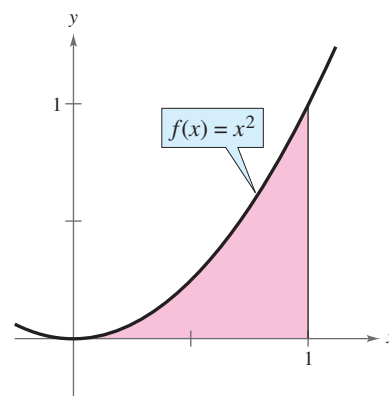


Figure 11.38

CHECKPOINT Now try Exercise 33.

Example 6 Finding the Area of a Region

Find the area of the region bounded by the graph of $f(x) = 3x - x^2$ and the x -axis between $x = 1$ and $x = 2$, as shown in Figure 11.39.

Solution

Begin by finding the dimensions of the rectangles.

$$\text{Width: } \frac{b-a}{n} = \frac{2-1}{n} = \frac{1}{n}$$

$$\begin{aligned} \text{Height: } f\left(a + \frac{(b-a)i}{n}\right) &= f\left(1 + \frac{i}{n}\right) \\ &= 3\left(1 + \frac{i}{n}\right) - \left(1 + \frac{i}{n}\right)^2 \\ &= 3 + \frac{3i}{n} - \left(1 + \frac{2i}{n} + \frac{i^2}{n^2}\right) \\ &= 2 + \frac{i}{n} - \frac{i^2}{n^2} \end{aligned}$$

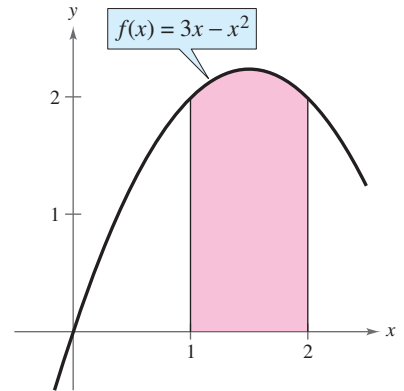


Figure 11.39

Next, approximate the area as the sum of the areas of n rectangles.

$$\begin{aligned} A &\approx \sum_{i=1}^n f\left(a + \frac{(b-a)i}{n}\right) \left(\frac{b-a}{n}\right) \\ &= \sum_{i=1}^n \left(2 + \frac{i}{n} - \frac{i^2}{n^2}\right) \left(\frac{1}{n}\right) \\ &= \frac{1}{n} \sum_{i=1}^n 2 + \frac{1}{n^2} \sum_{i=1}^n i - \frac{1}{n^3} \sum_{i=1}^n i^2 \\ &= \frac{1}{n}(2n) + \frac{1}{n^2} \left[\frac{n(n+1)}{2} \right] - \frac{1}{n^3} \left[\frac{n(n+1)(2n+1)}{6} \right] \\ &= 2 + \frac{n^2+n}{2n^2} - \frac{2n^3+3n^2+n}{6n^3} \\ &= 2 + \frac{1}{2} + \frac{1}{2n} - \frac{1}{3} - \frac{1}{2n} - \frac{1}{6n^2} \\ &= \frac{13}{6} - \frac{1}{6n^2} \end{aligned}$$

Finally, find the exact area by taking the limit as n approaches ∞ .

$$\begin{aligned} A &= \lim_{n \rightarrow \infty} \left(\frac{13}{6} - \frac{1}{6n^2} \right) \\ &= \frac{13}{6} \end{aligned}$$



Now try Exercise 39.

11.5 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Vocabulary Check

Fill in the blanks.

1. $\sum_{i=1}^n i = \underline{\hspace{2cm}}$ 2. $\sum_{i=1}^n i^3 = \underline{\hspace{2cm}}$

3. The exact $\underline{\hspace{2cm}}$ of a plane region R is given by the limit of the sum of n rectangles as n approaches ∞ .

In Exercises 1–8, evaluate the sum using the summation formulas and properties.

- | | |
|--------------------------------|-----------------------------------|
| 1. $\sum_{i=1}^{60} 7$ | 2. $\sum_{i=1}^{45} 3$ |
| 3. $\sum_{i=1}^{20} i^3$ | 4. $\sum_{i=1}^{30} i^2$ |
| 5. $\sum_{k=1}^{20} (k^3 + 2)$ | 6. $\sum_{k=1}^{50} (2k + 1)$ |
| 7. $\sum_{j=1}^{25} (j^2 + j)$ | 8. $\sum_{j=1}^{10} (j^3 - 3j^2)$ |

In Exercises 9–16, (a) rewrite the sum as a rational function $S(n)$. (b) Use $S(n)$ to complete the table. (c) Find $\lim_{n \rightarrow \infty} S(n)$.

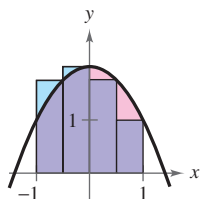
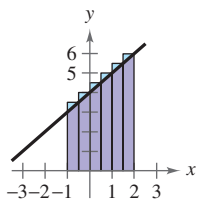
n	10^0	10^1	10^2	10^3	10^4
$S(n)$					

- | | |
|---|---|
| 9. $\sum_{i=1}^n \frac{i^3}{n^4}$ | 10. $\sum_{i=1}^n \frac{i}{n^2}$ |
| 11. $\sum_{i=1}^n \frac{3}{n^3} (1 + i^2)$ | 12. $\sum_{i=1}^n \frac{2i + 3}{n^2}$ |
| 13. $\sum_{i=1}^n \left(\frac{i^2}{n^3} + \frac{2}{n} \right) \left(\frac{1}{n} \right)$ | 14. $\sum_{i=1}^n \left[3 - 2 \left(\frac{i}{n} \right) \right] \left(\frac{1}{n} \right)$ |
| 15. $\sum_{i=1}^n \left[1 - \left(\frac{i}{n} \right)^2 \right] \left(\frac{1}{n} \right)$ | 16. $\sum_{i=1}^n \left[\frac{4}{n} + \left(\frac{2i}{n^2} \right) \right] \left(\frac{2i}{n} \right)$ |

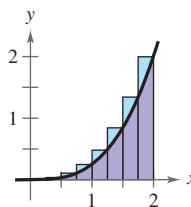
In Exercises 17–20, approximate the area of the region using the indicated number of rectangles of equal width.

17. $f(x) = x + 4$

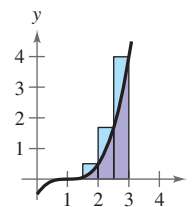
18. $f(x) = 2 - x^2$



19. $f(x) = \frac{1}{4}x^3$



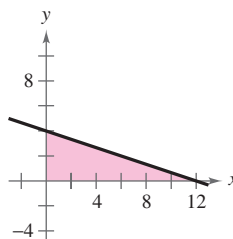
20. $f(x) = \frac{1}{2}(x - 1)^3$



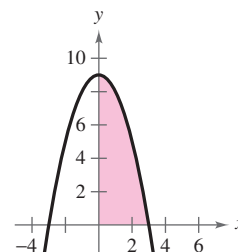
In Exercises 21–24, complete the table showing the approximate area of the region in the graph using n rectangles of equal width.

n	4	8	20	50
Approximate area				

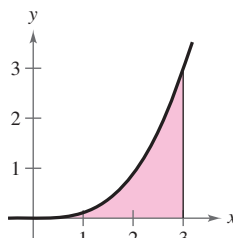
21. $f(x) = -\frac{1}{3}x + 4$



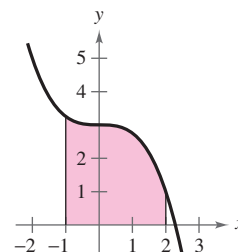
22. $f(x) = 9 - x^2$



23. $f(x) = \frac{1}{9}x^3$



24. $f(x) = 3 - \frac{1}{4}x^3$



In Exercises 25–32, complete the table using the function $f(x)$, over the specified interval $[a, b]$, to approximate the area of the region bounded by the graph of $y = f(x)$, the x -axis, and the vertical lines $x = a$ and $x = b$ using the indicated number of rectangles. Then find the exact area as $n \rightarrow \infty$.

n	4	8	20	50	100	∞
Area						

Function Interval

- 25. $f(x) = 2x + 5$ [0, 4]
- 26. $f(x) = 3x + 1$ [0, 4]
- 27. $f(x) = 16 - 2x$ [1, 5]
- 28. $f(x) = 20 - 2x$ [2, 6]
- 29. $f(x) = 9 - x^2$ [0, 2]
- 30. $f(x) = x^2 + 1$ [4, 6]
- 31. $f(x) = \frac{1}{2}x + 4$ [-1, 3]
- 32. $f(x) = \frac{1}{2}x + 1$ [-2, 2]

In Exercises 33–44, use the limit process to find the area of the region between the graph of the function and the x -axis over the specified interval.

Function Interval

- 33. $f(x) = 4x + 1$ [0, 1]
- 34. $f(x) = 3x + 2$ [0, 2]
- 35. $f(x) = -2x + 3$ [0, 1]
- 36. $f(x) = 3x - 4$ [2, 5]
- 37. $f(x) = 2 - x^2$ [-1, 1]
- 38. $f(x) = x^2 + 2$ [0, 1]
- 39. $g(x) = 8 - x^3$ [1, 2]
- 40. $g(x) = 64 - x^3$ [1, 4]
- 41. $g(x) = 2x - x^3$ [0, 1]
- 42. $g(x) = 4x - x^3$ [0, 2]
- 43. $f(x) = \frac{1}{4}(x^2 + 4x)$ [1, 4]
- 44. $f(x) = x^2 - x^3$ [-1, 1]

45. **Geometry** The boundaries of a parcel of land are two edges modeled by the coordinate axes and a stream modeled by the equation $y = (-3.0 \times 10^{-6})x^3 + 0.002x^2 - 1.05x + 400$. Use a graphing utility to graph the equation. Find the area of the property. Assume all distances are measured in feet.

46. **Geometry** The table shows the measurements (in feet) of a lot bounded by a stream and two straight roads that meet at right angles (see figure).

x	0	50	100	150	200	250	300
y	450	362	305	268	245	156	0

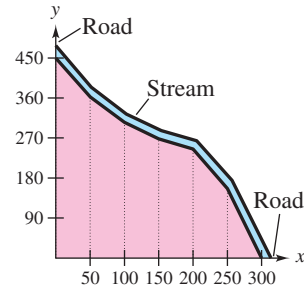


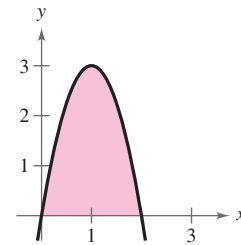
Figure for 46

- (a) Use the *regression* feature of a graphing utility to find a model of the form $y = ax^3 + bx^2 + cx + d$.
- (b) Use a graphing utility to plot the data and graph the model in the same viewing window.
- (c) Use the model in part (a) to estimate the area of the lot.

Synthesis

True or False? In Exercises 47 and 48, determine whether the statement is true or false. Justify your answer.

- 47. The sum of the first n positive integers is $n(n + 1)/2$.
- 48. The exact area of a region is given by the limit of the sum of n rectangles as n approaches 0.
- 49. **Writing** Describe the process of finding the area of a region bounded by the graph of a nonnegative, continuous function f , the x -axis, and the vertical lines $x = a$ and $x = b$.
- 50. **Think About It** Determine which value best approximates the area of the region shown in the graph. (Make your selection on the basis of the sketch of the region and *not* by performing any calculations.)
 (a) -2 (b) 1 (c) 4 (d) 6 (e) 9



Skills Review

In Exercises 51 and 52, solve the equation.

- 51. $2 \tan x = \tan 2x$
- 52. $\cos 2x - 3 \sin x = 2$

In Exercises 53–56, use the vectors $\mathbf{u} = \langle 4, -5 \rangle$ and $\mathbf{v} = \langle -1, -2 \rangle$ to find the indicated quantity.

- 53. $(\mathbf{u} \cdot \mathbf{v})\mathbf{u}$
- 54. $3\mathbf{u} \cdot \mathbf{v}$
- 55. $\|\mathbf{v}\| - 2$
- 56. $\|\mathbf{u}\|^2 - \|\mathbf{v}\|^2$

What Did You Learn?

Key Terms

limit, p. 781	one-sided limit, p. 795	slope of a graph, p. 803
direct substitution, p. 785	tangent line, p. 801	limits at infinity, p. 811
dividing out technique, p. 791	secant line, p. 803	converge, p. 815
indeterminate form, p. 792	difference quotient, p. 803	diverge, p. 815
rationalizing technique, p. 793		

Key Concepts

11.1 ■ Conditions under which limits do not exist

The limit of $f(x)$ as $x \rightarrow c$ does not exist if any of the following conditions is true.

- $f(x)$ approaches a different number from the right side of c than it approaches from the left side of c .
- $f(x)$ increases or decreases without bound as $x \rightarrow c$.
- $f(x)$ oscillates between two fixed values as $x \rightarrow c$.

11.1 ■ Use basic limits and properties of limits

Let b and c be real numbers, let n be a positive integer, and let f and g be functions with the limits $\lim_{x \rightarrow c} f(x) = L$ and $\lim_{x \rightarrow c} g(x) = K$.

- $\lim_{x \rightarrow c} b = b$
- $\lim_{x \rightarrow c} x = c$
- $\lim_{x \rightarrow c} x^n = c^n$
- $\lim_{x \rightarrow c} \sqrt[n]{x} = \sqrt[n]{c}$, n even, $c > 0$
- $\lim_{x \rightarrow c} [bf(x)] = bL$
- $\lim_{x \rightarrow c} [f(x) \pm g(x)] = L \pm K$
- $\lim_{x \rightarrow c} [f(x)g(x)] = LK$
- $\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \frac{L}{K}$, $K \neq 0$
- $\lim_{x \rightarrow c} [f(x)]^n = L^n$

11.1 ■ Limits of polynomial and rational functions

- If p is a polynomial function and c is a real number, then $\lim_{x \rightarrow c} p(x) = p(c)$.
- If r is a rational function given by $r(x) = p(x)/q(x)$, and c is a real number such that $q(c) \neq 0$, then

$$\lim_{x \rightarrow c} r(x) = r(c) = \frac{p(c)}{q(c)}.$$

11.2 ■ Determine the existence of limits

If f is a function and c and L are real numbers, then $\lim_{x \rightarrow c} f(x) = L$ if and only if both the left and right limits exist and are equal to L .

11.3 ■ Find derivatives of functions

The derivative of f at x is given by

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

provided this limit exists.

11.4 ■ Evaluate limits at infinity

- If r is a positive real number, then $\lim_{x \rightarrow \infty} (1/x^r) = 0$.

If x^r is defined when $x < 0$, then $\lim_{x \rightarrow -\infty} (1/x^r) = 0$.

- Consider the rational function $f(x) = N(x)/D(x)$, where $N(x) = a_n x^n + \dots + a_0$ and $D(x) = b_m x^m + \dots + b_0$. The limit of $f(x)$ as x approaches positive or negative infinity is as follows. Note that if $n > m$, the limit does not exist.

$$\lim_{x \rightarrow \pm\infty} f(x) = \begin{cases} 0, & n < m \\ \frac{a_n}{b_m}, & n = m \end{cases}$$

11.4 ■ Find limits of sequences

Let f be a function of a real variable such that $\lim_{x \rightarrow \infty} f(x) = L$. If $\{a_n\}$ is a sequence such that $f(n) = a_n$ for every positive integer n , then $\lim_{n \rightarrow \infty} a_n = L$.

11.5 ■ Use summation formulas and properties

- $\sum_{i=1}^n c = cn$, c is a constant.
- $\sum_{i=1}^n i = \frac{n(n+1)}{2}$
- $\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$
- $\sum_{i=1}^n i^3 = \frac{n^2(n+1)^2}{4}$
- $\sum_{i=1}^n (a_i \pm b_i) = \sum_{i=1}^n a_i \pm \sum_{i=1}^n b_i$
- $\sum_{i=1}^n ka_i = k \sum_{i=1}^n a_i$, k is a constant.

11.5 ■ Find the area of a plane region

Let f be continuous and nonnegative on the interval $[a, b]$. The area A of the region bounded by the graph of f , the x -axis, and the vertical lines $x = a$ and $x = b$ is given by

$$A = \lim_{n \rightarrow \infty} \sum_{i=1}^n f\left(\underbrace{a + \frac{(b-a)i}{n}}_{\text{Height}}\right) \underbrace{\left(\frac{b-a}{n}\right)}_{\text{Width}}.$$

Review Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

11.1 In Exercises 1–4, complete the table and use the result to estimate the limit numerically. Determine whether or not the limit can be reached.

1. $\lim_{x \rightarrow 3} (6x - 1)$

x	2.9	2.99	2.999	3	3.001	3.01	3.1
$f(x)$?			

2. $\lim_{x \rightarrow 2} \frac{x - 2}{3x^2 - 4x - 4}$

x	1.9	1.99	1.999	2	2.001	2.01	2.1
$f(x)$?			

3. $\lim_{x \rightarrow 0} \frac{1 - e^{-x}}{x}$

x	-0.1	-0.01	-0.001	0
$f(x)$?

x	0.001	0.01	0.1
$f(x)$			

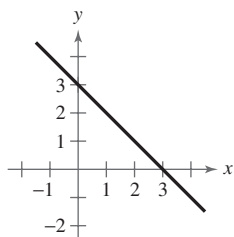
4. $\lim_{x \rightarrow 0} \frac{\ln(1 - x)}{x}$

x	-0.1	-0.01	-0.001	0
$f(x)$?

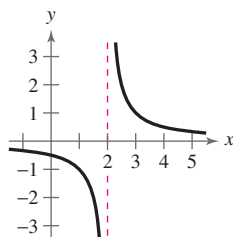
x	0.001	0.01	0.1
$f(x)$			

In Exercises 5–8, use the graph to find the limit (if it exists). If the limit does not exist, explain why.

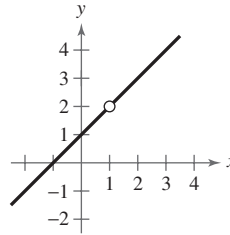
5. $\lim_{x \rightarrow 1} (3 - x)$



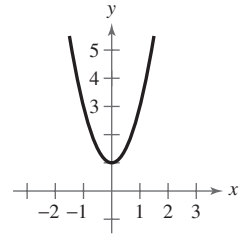
6. $\lim_{x \rightarrow 2} \frac{1}{x - 2}$



7. $\lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1}$



8. $\lim_{x \rightarrow -1} (2x^2 + 1)$



In Exercises 9 and 10, use the given information to evaluate each limit.

9. $\lim_{x \rightarrow c} f(x) = 4, \lim_{x \rightarrow c} g(x) = 5$

- (a) $\lim_{x \rightarrow c} [f(x)]^3$
- (b) $\lim_{x \rightarrow c} [3f(x) - g(x)]$
- (c) $\lim_{x \rightarrow c} [f(x)g(x)]$
- (d) $\lim_{x \rightarrow c} \frac{f(x)}{g(x)}$

10. $\lim_{x \rightarrow c} f(x) = 27, \lim_{x \rightarrow c} g(x) = 12$

- (a) $\lim_{x \rightarrow c} \sqrt[3]{f(x)}$
- (b) $\lim_{x \rightarrow c} \frac{f(x)}{18}$
- (c) $\lim_{x \rightarrow c} [f(x)g(x)]$
- (d) $\lim_{x \rightarrow c} [f(x) - 2g(x)]$

In Exercises 11–24, find the limit by direct substitution.

- 11. $\lim_{x \rightarrow 4} (\frac{1}{2}x + 3)$
- 12. $\lim_{x \rightarrow 3} (5x - 4)$
- 13. $\lim_{x \rightarrow 2} (5x - 3)(3x + 5)$
- 14. $\lim_{x \rightarrow -2} (5 - 2x - x^2)$
- 15. $\lim_{t \rightarrow 3} \frac{t^2 + 1}{t}$
- 16. $\lim_{x \rightarrow 2} \frac{3x + 5}{5x - 3}$
- 17. $\lim_{x \rightarrow -2} \sqrt[3]{4x}$
- 18. $\lim_{x \rightarrow -1} \sqrt{5 - x}$
- 19. $\lim_{x \rightarrow \pi} \sin 3x$
- 20. $\lim_{x \rightarrow 0} \tan x$
- 21. $\lim_{x \rightarrow -1} 2e^x$
- 22. $\lim_{x \rightarrow 4} \ln x$
- 23. $\lim_{x \rightarrow -1/2} \arcsin x$
- 24. $\lim_{x \rightarrow 0} \arctan x$

11.2 In Exercises 25–36, find the limit (if it exists). Use a graphing utility to verify your result graphically.

- 25. $\lim_{t \rightarrow -2} \frac{t + 2}{t^2 - 4}$
- 26. $\lim_{t \rightarrow 3} \frac{t^2 - 9}{t - 3}$
- 27. $\lim_{x \rightarrow 5} \frac{x - 5}{x^2 + 5x - 50}$
- 28. $\lim_{x \rightarrow -1} \frac{x + 1}{x^2 - 5x - 6}$
- 29. $\lim_{x \rightarrow -2} \frac{x^2 - 4}{x^3 + 8}$
- 30. $\lim_{x \rightarrow 4} \frac{x^3 - 64}{x^2 - 16}$

$$31. \lim_{x \rightarrow -1} \frac{\frac{1}{x+2} - 1}{x+1}$$

$$32. \lim_{x \rightarrow 0} \frac{\frac{1}{x+1} - 1}{x}$$

$$33. \lim_{u \rightarrow 0} \frac{\sqrt{4+u} - 2}{u}$$

$$34. \lim_{v \rightarrow 0} \frac{\sqrt{v+9} - 3}{v}$$

$$35. \lim_{x \rightarrow 5} \frac{\sqrt{x-1} - 2}{x-5}$$

$$36. \lim_{x \rightarrow 1} \frac{\sqrt{3} - \sqrt{x+2}}{1-x}$$

Graphical and Numerical Analysis In Exercises 37–44, (a) graphically approximate the limit (if it exists) by using a graphing utility to graph the function and (b) numerically approximate the limit (if it exists) by using the *table* feature of a graphing utility to create a table.

$$37. \lim_{x \rightarrow 3} \frac{x-3}{x^2-9}$$

$$38. \lim_{x \rightarrow 4} \frac{4-x}{16-x^2}$$

$$39. \lim_{x \rightarrow 0} e^{-2/x}$$

$$40. \lim_{x \rightarrow 0} e^{-4/x^2}$$

$$41. \lim_{x \rightarrow 0} \frac{\sin 4x}{2x}$$

$$42. \lim_{x \rightarrow 0} \frac{\tan 2x}{x}$$

$$43. \lim_{x \rightarrow 1^+} \frac{\sqrt{2x+1} - \sqrt{3}}{x-1}$$

$$44. \lim_{x \rightarrow 1^+} \frac{1 - \sqrt{x}}{x-1}$$

In Exercises 45–52, graph the function. Determine the limit (if it exists) by evaluating the corresponding one-sided limits.

$$45. \lim_{x \rightarrow 3} \frac{|x-3|}{x-3}$$

$$46. \lim_{x \rightarrow 8} \frac{|8-x|}{8-x}$$

$$47. \lim_{x \rightarrow 2} \frac{2}{x^2-4}$$

$$48. \lim_{x \rightarrow -3} \frac{1}{x^2+9}$$

$$49. \lim_{x \rightarrow 5} \frac{|x-5|}{x-5}$$

$$50. \lim_{x \rightarrow -2} \frac{|x+2|}{x+2}$$

$$51. \lim_{x \rightarrow 2} f(x) \text{ where } f(x) = \begin{cases} 5-x, & x \leq 2 \\ x^2-3, & x > 2 \end{cases}$$

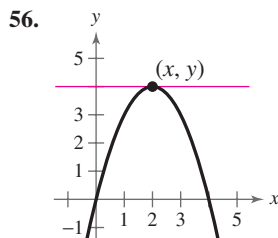
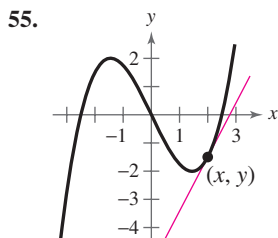
$$52. \lim_{x \rightarrow 0} f(x) \text{ where } f(x) = \begin{cases} x-6, & x \geq 0 \\ x^2-4, & x < 0 \end{cases}$$

f In Exercises 53 and 54, find $\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$.

$$53. f(x) = 3x - x^2$$

$$54. f(x) = x^2 - 5x - 2$$

11.3 In Exercises 55 and 56, approximate the slope of the tangent line to the graph at the point (x, y) .



In Exercises 57–62, use a graphing utility to graph the function and the tangent line at the point $(2, f(2))$. Use the graph to approximate the slope of the tangent line.

$$57. f(x) = x^2 - 2x$$

$$58. f(x) = 6 - x^2$$

$$59. f(x) = \sqrt{x+2}$$

$$60. f(x) = \sqrt{x^2+5}$$

$$61. f(x) = \frac{6}{x-4}$$

$$62. f(x) = \frac{1}{3-x}$$

In Exercises 63–66, find a formula for the slope of the graph of f at the point $(x, f(x))$. Then use it to find the slopes at the two specified points.

$$63. f(x) = x^2 - 4x$$

$$64. f(x) = \frac{1}{4}x^4$$

$$(a) (0, 0)$$

$$(a) (-2, 4)$$

$$(b) (5, 5)$$

$$(b) (1, \frac{1}{4})$$

$$65. f(x) = \frac{4}{x-6}$$

$$66. f(x) = \sqrt{x}$$

$$(a) (7, 4)$$

$$(a) (1, 1)$$

$$(b) (8, 2)$$

$$(b) (4, 2)$$

f In Exercises 67–78, find the derivative of the function.

$$67. f(x) = 5$$

$$68. g(x) = -3$$

$$69. h(x) = 5 - \frac{1}{2}x$$

$$70. f(x) = 3x$$

$$71. g(x) = 2x^2 - 1$$

$$72. f(x) = -x^3 + 4x$$

$$73. f(t) = \sqrt{t+5}$$

$$74. g(t) = \sqrt{t-3}$$

$$75. g(s) = \frac{4}{s+5}$$

$$76. g(t) = \frac{6}{5-t}$$

$$77. g(x) = \frac{1}{\sqrt{x+4}}$$

$$78. f(x) = \frac{1}{\sqrt{12-x}}$$

11.4 In Exercises 79–86, find the limit (if it exists). If the limit does not exist, explain why. Use a graphing utility to verify your result graphically.

$$79. \lim_{x \rightarrow \infty} \frac{4x}{2x-3}$$

$$80. \lim_{x \rightarrow \infty} \frac{7x}{14x+2}$$

$$81. \lim_{x \rightarrow -\infty} \frac{2x}{x^2-25}$$

$$82. \lim_{x \rightarrow -\infty} \frac{3x}{(1-x)^3}$$

$$83. \lim_{x \rightarrow \infty} \frac{x^2}{2x+3}$$

$$84. \lim_{y \rightarrow \infty} \frac{3y^4}{y^2+1}$$

$$85. \lim_{x \rightarrow \infty} \left[\frac{x}{(x-2)^2} + 3 \right]$$

$$86. \lim_{x \rightarrow \infty} \left[2 - \frac{2x^2}{(x+1)^2} \right]$$

In Exercises 87–92, write the first five terms of the sequence and find the limit of the sequence (if it exists). If the limit does not exist, explain why. Assume n begins with 1.

$$87. a_n = \frac{2n-3}{5n+4}$$

$$88. a_n = \frac{2n}{n^2+1}$$

89. $a_n = \frac{(-1)^n}{n^3}$

90. $a_n = \frac{(-1)^{n+1}}{n}$

91. $a_n = \frac{1}{2n^2}[3 - 2n(n + 1)]$

92. $a_n = \left(\frac{2}{n}\right)\left\{n + \frac{2}{n}\left[\frac{n(n-1)}{2} - n\right]\right\}$

11.5 In Exercises 93 and 94, (a) use the summation formulas and properties to rewrite the sum as a rational function $S(n)$. (b) Use $S(n)$ to complete the table. (c) Find $\lim_{n \rightarrow \infty} S(n)$.

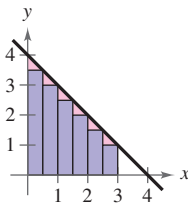
n	10^0	10^1	10^2	10^3	10^4
$S(n)$					

93. $\sum_{i=1}^n \left(\frac{4i^2}{n^2} - \frac{i}{n}\right)\left(\frac{1}{n}\right)$

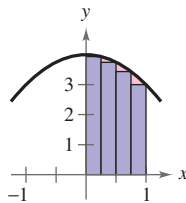
94. $\sum_{i=1}^n \left[4 - \left(\frac{3i}{n}\right)^2\right]\left(\frac{3i}{n^2}\right)$

In Exercises 95 and 96, approximate the area of the region using the indicated number of rectangles of equal width.

95. $f(x) = 4 - x$



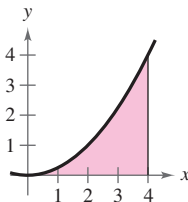
96. $f(x) = 4 - x^2$



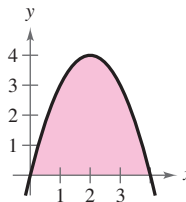
In Exercises 97 and 98, complete the table showing the approximate area of the region in the graph using n rectangles of equal width.

n	4	8	20	50
Approximate area				

97. $f(x) = \frac{1}{4}x^2$



98. $f(x) = 4x - x^2$



In Exercises 99–106, use the limit process to find the area of the region between the graph of the function and the x -axis over the specified interval.

Function

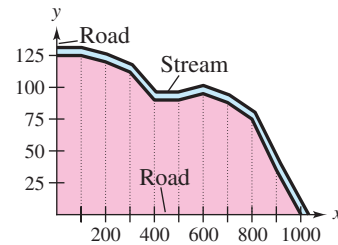
Interval

- 99. $f(x) = 10 - x$ [0, 10]
- 100. $f(x) = 2x - 6$ [3, 6]
- 101. $f(x) = x^2 + 4$ [-1, 2]
- 102. $f(x) = 8(x - x^2)$ [0, 1]
- 103. $f(x) = x^3 + 1$ [0, 2]
- 104. $f(x) = 1 - x^3$ [-3, -1]
- 105. $f(x) = 3(x^3 - x^2)$ [1, 3]
- 106. $f(x) = 5 - (x + 2)^2$ [-2, 0]

107. Geometry The table shows the measurements (in feet) of a lot bounded by a stream and two straight roads that meet at right angles (see figure).

x	0	100	200	300	400	500
y	125	125	120	112	90	90

x	600	700	800	900	1000
y	95	88	75	35	0



- (a) Use the *regression* feature of a graphing utility to find a model of the form $y = ax^3 + bx^2 + cx + d$.
- (b) Use a graphing utility to plot the data and graph the model in the same viewing window.
- (c) Use the model in part (a) to estimate the area of the lot.

Synthesis

True or False? In Exercises 108 and 109, determine whether the statement is true or false. Justify your answer.

- 108. The limit of the sum of two functions is the sum of the limits of the two functions.
- 109. If the degree of the numerator $N(x)$ of a rational function $f(x) = N(x)/D(x)$ is greater than the degree of its denominator $D(x)$, then the limit of the rational function as x approaches ∞ is 0.

110. Writing Write a short paragraph explaining several reasons why the limit of a function may not exist.

11 Chapter Test

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Take this test as you would take a test in class. After you are finished, check your work against the answers given in the back of the book.

In Exercises 1–3, use a graphing utility to graph the function and approximate the limit (if it exists). Then find the limit (if it exists) algebraically by using appropriate techniques.

$$1. \lim_{x \rightarrow -2} \frac{x^2 - 1}{2x} \quad 2. \lim_{x \rightarrow 1} \frac{-x^2 + 5x - 3}{1 - x} \quad 3. \lim_{x \rightarrow 5} \frac{\sqrt{x} - 2}{x - 5}$$

In Exercises 4 and 5, use a graphing utility to graph the function and approximate the limit. Write an approximation that is accurate to four decimal places. Then create a table to verify your limit numerically.

$$4. \lim_{x \rightarrow 0} \frac{\sin 3x}{x} \quad 5. \lim_{x \rightarrow 0} \frac{e^{2x} - 1}{x}$$

6. Find a formula for the slope of the graph of f at the point $(x, f(x))$. Then use it to find the slope at the specified point.

$$(a) f(x) = 3x^2 - 5x - 2, (2, 0) \quad (b) f(x) = 2x^3 + 6x, (-1, -8)$$

In Exercises 7–9, find the derivative of the function.

$$7. f(x) = 5 - \frac{2}{5}x \quad 8. f(x) = 2x^2 + 4x - 1 \quad 9. f(x) = \frac{1}{x + 3}$$

In Exercises 10–12, find the limit (if it exists). If the limit does not exist, explain why. Use a graphing utility to verify your result graphically.

$$10. \lim_{x \rightarrow \infty} \frac{6}{5x - 1} \quad 11. \lim_{x \rightarrow \infty} \frac{1 - 3x^2}{x^2 - 5} \quad 12. \lim_{x \rightarrow -\infty} \frac{x^2}{3x + 2}$$

In Exercises 13 and 14, write the first five terms of the sequence and find the limit of the sequence (if it exists). If the limit does not exist, explain why. Assume n begins with 1.

$$13. a_n = \frac{n^2 + 3n - 4}{2n^2 + n - 2} \quad 14. a_n = \frac{1 + (-1)^n}{n}$$

15. Approximate the area of the region bounded by the graph of $f(x) = 8 - 2x^2$ shown at the right using the indicated number of rectangles of equal width.

In Exercises 16 and 17, use the limit process to find the area of the region between the graph of the function and the x -axis over the specified interval.

$$16. f(x) = x + 2; \text{ interval: } [-2, 2] \quad 17. f(x) = 3 - x^2; \text{ interval: } [-1, 1]$$

18. The table shows the height of a space shuttle during its first 5 seconds of motion.
- Use the *regression* feature of a graphing utility to find a quadratic model $y = ax^2 + bx + c$ for the data.
 - The value of the derivative of the model is the rate of change of height with respect to time, or the velocity, at that instant. Find the velocity of the shuttle after 5 seconds.

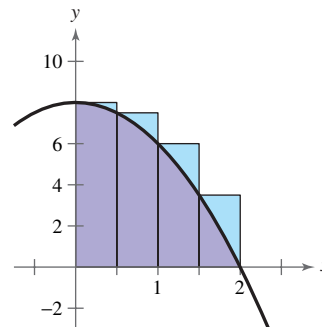


Figure for 15



Time (seconds), x	Height (feet), y
0	0
1	1
2	23
3	60
4	115
5	188

Table for 18

10–11 Cumulative Test

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Take this test to review the material in Chapters 10 and 11. After you are finished, check your work against the answers given in the back of the book.

In Exercises 1 and 2, find the coordinates of the point.

- The point is located six units behind the yz -plane, one unit to the right of the xz -plane, and three units above the xy -plane.
- The point is located on the y -axis, four units to the left of the xz -plane.
- Find the distance between the points $(-2, 3, -6)$ and $(4, -5, 1)$.
- Find the lengths of the sides of the right triangle at the right. Show that these lengths satisfy the Pythagorean Theorem.
- Find the coordinates of the midpoint of the line segment joining $(3, 4, -1)$ and $(-5, 0, 2)$.
- Find an equation of the sphere for which the endpoints of a diameter are $(0, 0, 0)$ and $(4, 4, 8)$.
- Sketch the graph of the equation $(x - 2)^2 + (y + 1)^2 + z^2 = 4$, and then sketch the xy -trace and the yz -trace.
- For the vectors $\mathbf{u} = \langle 2, -6, 0 \rangle$ and $\mathbf{v} = \langle -4, 5, 3 \rangle$, find $\mathbf{u} \cdot \mathbf{v}$ and $\mathbf{u} \times \mathbf{v}$.

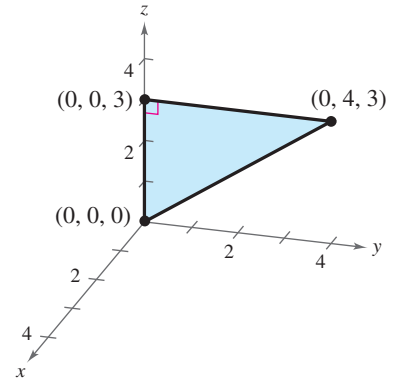


Figure for 4

In Exercises 9–11, determine whether \mathbf{u} and \mathbf{v} are orthogonal, parallel, or neither.

- | | | |
|--|---|--|
| 9. $\mathbf{u} = \langle 4, 4, 0 \rangle$
$\mathbf{v} = \langle 0, -8, 6 \rangle$ | 10. $\mathbf{u} = \langle 4, -2, 10 \rangle$
$\mathbf{v} = \langle -2, 6, 2 \rangle$ | 11. $\mathbf{u} = \langle -1, 6, -3 \rangle$
$\mathbf{v} = \langle 3, -18, 9 \rangle$ |
|--|---|--|

- Find the volume of the parallelepiped with the vertices $A(1, 3, 2)$, $B(3, 4, 2)$, $C(3, 2, 2)$, $D(1, 1, 2)$, $E(1, 3, 5)$, $F(3, 4, 5)$, $G(3, 2, 5)$, and $H(1, 1, 5)$.
- Find sets of (a) parametric equations and (b) symmetric equations for the line passing through the points $(-2, 3, 0)$ and $(5, 8, 25)$.
- Find the parametric form of the equation of the line passing through the point $(-1, 2, 0)$ and perpendicular to $2x - 4y + z = 8$.
- Find an equation of the plane passing through the points $(0, 0, 0)$, $(-2, 3, 0)$, and $(5, 8, 25)$.
- Label the intercepts and sketch the graph of the plane given by $3x - 6y - 12z = 24$.
- Find the distance between the point $(0, 0, 25)$ and the plane $2x - 5y + z = 10$.
- A plastic wastebasket has the shape and dimensions shown in the figure. In fabricating a mold for making the wastebasket, it is necessary to know the angle between two adjacent sides. Find the angle.

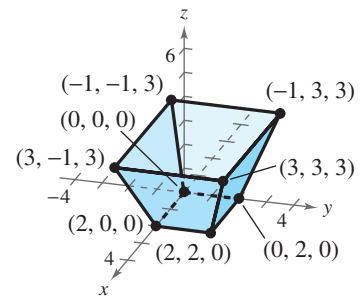


Figure for 18

In Exercises 19–27, find the limit (if it exists). If the limit does not exist, explain why. Use a graphing utility to verify your result graphically.

- | | | |
|--|---|---|
| 19. $\lim_{x \rightarrow 4} (5x - x^2)$ | 20. $\lim_{x \rightarrow -2^+} \frac{x + 2}{x^2 + x - 2}$ | 21. $\lim_{x \rightarrow 7} \frac{x - 7}{x^2 - 49}$ |
| 22. $\lim_{x \rightarrow 0} \frac{\sqrt{x + 4} - 2}{x}$ | 23. $\lim_{x \rightarrow 4^-} \frac{ x - 4 }{x - 4}$ | 24. $\lim_{x \rightarrow 0} \sin\left(\frac{\pi}{x}\right)$ |
| 25. $\lim_{x \rightarrow 0} \frac{\frac{1}{x + 2} - \frac{1}{2}}{x}$ | 26. $\lim_{x \rightarrow 0} \frac{\sqrt{x + 1} - 1}{x}$ | 27. $\lim_{x \rightarrow 2^-} \frac{x - 2}{x^2 - 4}$ |

In Exercises 28–31, find a formula for the slope of f at the point $(x, f(x))$. Then use it to find the slope at the specified point.

28. $f(x) = 4 - x^2$, $(0, 4)$

29. $f(x) = \sqrt{x + 3}$, $(-2, 1)$

30. $f(x) = \frac{1}{x + 3}$, $(1, \frac{1}{4})$

31. $f(x) = x^2 - x$, $(1, 0)$

In Exercises 32–37, find the limit (if it exists). If the limit does not exist, explain why. Use a graphing utility to verify your result graphically.

32. $\lim_{x \rightarrow \infty} \frac{x^3}{x^2 - 9}$

33. $\lim_{x \rightarrow \infty} \frac{3 - 7x}{x + 4}$

34. $\lim_{x \rightarrow \infty} \frac{3x^2 + 1}{x^2 + 4}$

35. $\lim_{x \rightarrow \infty} \frac{2x}{x^2 + 3x - 2}$

36. $\lim_{x \rightarrow \infty} \frac{3 - x}{x^2 + 1}$

37. $\lim_{x \rightarrow \infty} \frac{3 + 4x - x^3}{2x^2 + 3}$

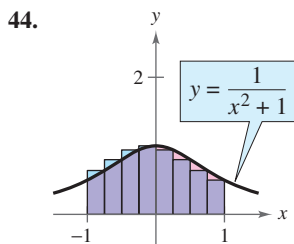
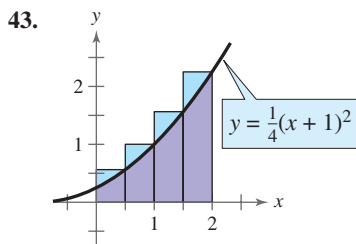
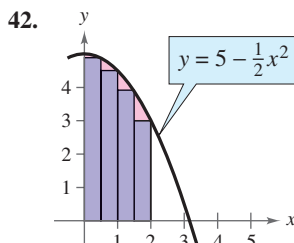
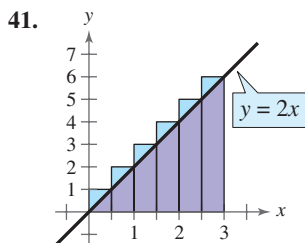
In Exercises 38–40, evaluate the sum using the summation formulas and properties.

38. $\sum_{i=1}^{50} (1 - i^2)$

39. $\sum_{k=1}^{20} (3k^2 - 2k)$

40. $\sum_{i=1}^{40} (12 + i^3)$

In Exercises 41–44, approximate the area of the region using the indicated number of rectangles of equal width.



In Exercises 45–50, use the limit process to find the area of the region between the graph of the function and the x -axis over the specified interval.

45. $f(x) = x + 2$

Interval: $[0, 1]$

46. $f(x) = 8 - 2x$

Interval: $[-4, 4]$

47. $f(x) = 2x + 5$

Interval: $[-1, 3]$

48. $f(x) = x^2 + 1$

Interval: $[0, 4]$

49. $f(x) = 4 - x^2$

Interval: $[0, 2]$

50. $f(x) = 1 - x^3$

Interval: $[0, 1]$

Proofs in Mathematics

Many of the proofs of the definitions and properties presented in this chapter are beyond the scope of this text. Included below are simple proofs for the limit of a power function and the limit of a polynomial function.

Limit of a Power Function (p. 785)

$\lim_{x \rightarrow c} x^n = c^n$, c is a real number and n is a positive integer.

Proof

$$\begin{aligned}
 \lim_{x \rightarrow c} x^n &= \lim_{x \rightarrow c} \underbrace{(x \cdot x \cdot x \cdot \dots \cdot x)}_{n \text{ factors}} \\
 &= \lim_{x \rightarrow c} x \cdot \lim_{x \rightarrow c} x \cdot \lim_{x \rightarrow c} x \cdot \dots \cdot \lim_{x \rightarrow c} x && \text{Product Property of Limits} \\
 &= \underbrace{c \cdot c \cdot c \cdot \dots \cdot c}_{n \text{ factors}} && \text{Limit of the identity function} \\
 &= c^n && \text{Exponential form}
 \end{aligned}$$

Proving Limits

To prove most of the definitions and properties from this chapter, you must use the *formal definition* of a limit. This definition is called the *epsilon-delta definition* and was first introduced by Karl Weierstrass (1815–1897). If you go on to take a course in calculus, you will use this definition of a limit extensively.

Limit of a Polynomial Function (p. 787)

If p is a polynomial function and c is a real number, then

$$\lim_{x \rightarrow c} p(x) = p(c).$$

Proof

Let p be a polynomial function such that

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_2 x^2 + a_1 x + a_0.$$

Because a polynomial function is the sum of monomial functions, you can write the following.

$$\begin{aligned}
 \lim_{x \rightarrow c} p(x) &= \lim_{x \rightarrow c} (a_n x^n + a_{n-1} x^{n-1} + \dots + a_2 x^2 + a_1 x + a_0) \\
 &= \lim_{x \rightarrow c} a_n x^n + \lim_{x \rightarrow c} a_{n-1} x^{n-1} + \dots + \lim_{x \rightarrow c} a_2 x^2 + \lim_{x \rightarrow c} a_1 x + \lim_{x \rightarrow c} a_0 \\
 &= a_n c^n + a_{n-1} c^{n-1} + \dots + a_2 c^2 + a_1 c + a_0 && \text{Scalar Multiple Property of Limits and limit of a power function} \\
 &= p(c) && p \text{ evaluated at } c
 \end{aligned}$$

Progressive Summary (Chapters 3–11)

This chart outlines the topics that have been covered so far in this text. Progressive summary charts appear after Chapters 2, 3, 6, 9, and 11. In each progressive summary, new topics encountered for the first time appear in red.

Transcendental Functions	Systems and Series	Other Topics																																										
Exponential, Logarithmic, Trigonometric, Inverse Trigonometric	Systems, Sequences, Series	Conics, Parametric and Polar Equations, 3-space, Limits																																										
<p>■ Rewriting</p> <p>Exponential form \leftrightarrow Logarithmic form Condense/expand logarithmic functions Simplify trigonometric expressions Prove trigonometric identities Use conversion formulas Operations with vectors Powers of roots and complex numbers</p>	<p>■ Rewriting</p> <p>Row operations for systems of equations Partial fraction decomposition Operations with matrices Matrix form of a system of equations nth term of a sequence Summation form of a series</p>	<p>■ Rewriting</p> <p>Standard forms of conics Eliminate parameters Rectangular form \leftrightarrow Parametric form Rectangular form \leftrightarrow Polar form Rationalize numerator Difference quotient</p>																																										
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